

# An application of the sum–product phenomenon to sets having no solutions of several linear equations \*

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## Abstract

We prove that for an arbitrary  $\kappa < \frac{5}{31}$  any subset of  $\mathbb{F}_p$  avoiding  $t$  linear equations with three variables has size less than  $O(p/t^\kappa)$ . We also find several applications to problems about so-called non-averaging sets, number of collinear triples and mixed energies.

## 1 Introduction

Let  $p$  be a prime number,  $\mathbb{F}_p$  be the finite field and  $A \subseteq \mathbb{F}_p$  be a set. Consider a linear equation

$$c_1x_1 + \cdots + c_kx_k = b, \quad (1)$$

where  $k \geq 3$  and  $c_1, \dots, c_k \neq 0$ . We say that our set  $A$  *avoids equation* (1) if there are no tuples  $(x_1, \dots, x_k) \in A^k$  satisfying (1). Sets avoiding linear equations is a well-known subject of Additive Combinatorics and Number Theory, see, e.g., classical papers [11, 12] about this question. It is known that if  $b = 0$  and  $c_1 + \cdots + c_k = 0$ , then  $|A| = o(p)$  as  $p \rightarrow \infty$  but in the other cases one can easily construct a set of positive density avoiding (1). In this paper we have to deal with the case  $k = 3$  but instead of one equation we consider several, say,  $t$  of them. Such problems are considered in articles [12], [4], for example. For us the basic question is the following: is it true that  $|A| = o_t(p)$  as  $t \rightarrow \infty$  (and  $p \rightarrow \infty$  of course)? Notice that we do not require  $b = 0$  or  $c_1 + c_2 + c_3 = 0$ . It turns out that the answer is positive and the problem is connected with the sum–product phenomenon, see, e.g., [22]. Let us formulate a special but an useful case of the main result of this paper (our general Theorem 28 is contained in section 6 below).

**Theorem 1** *Let  $A \subseteq \mathbb{F}_p$  be a set,  $|A| \gg p^{39/47}$ . Suppose that  $A$  avoids  $t$  equations of the form*

$$x_1 + a_jx_2 + b_jx_3 = b_j, \quad (2)$$

*where all  $a_j, b_j$  are nonzero and  $(a_i, b_i) \neq (a_j, b_j)$  for  $i \neq j$ ,  $i, j = 1, 2, \dots, t$ . Then for any  $\kappa < \frac{5}{31}$  one has*

$$|A| = O\left(\frac{p}{t^\kappa}\right). \quad (3)$$

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In another direction, there is a set  $A \subseteq \mathbb{F}_p$  avoiding  $t$  linear equations of form (2) such that

$$|A| \gg \frac{p}{t^{1/2}}. \quad (4)$$

Actually, we prove that  $\kappa$  in Theorem 1 can be doubled in many cases, see section 6, so the power of  $t$  is between  $10/31 > 0.3225$  and  $0.5$  for such wide class of equations. The author thinks that  $0.3225$  can be improved slightly but he does not believe that this constant can be replaced by something strictly greater than  $1/3$ , at least it requires some new ideas, would imply a considerable progress in the area and seems unattainable at the moment (see Example 11 in section 3).

The method of the proof is based on precise incidences results from [9] and some applications of these results from [1], [7], [10]. Usually, theorems of such a sort have to deal with small subsets of  $\mathbb{F}_p$ . Considering a dual set, that is, the *spectrum* of a set or, in other words, the set of large exponential sums, see section 4, we show that these results are applicable sometimes for *large* subsets of  $\mathbb{F}_p$ , exactly as in (3), (4). In particular, we prove the following fact, which is interesting in its own right: the spectrum always has small multiplicative energy, see Theorem 20 below.

The simplest example of system (2) can be obtained if one consider a multiplicative subgroup  $\Gamma \subseteq \mathbb{F}_p \setminus \{0\}$  and take just one linear equation  $\gamma = \alpha s_1 + \beta s_2$ , where  $\alpha, \beta \neq 0$  are fixed,  $\gamma \in \Gamma$  and  $s_1, s_2$  belong to  $\Gamma$ . Then this equation generates  $|\Gamma|^2$  another equations  $x = \alpha \gamma' y + \beta \gamma'' z$ , where  $\gamma', \gamma'' \in \Gamma$  and thus can be studied by the methods of our paper. Another nontrivial example is given by so-called *collinear triples* of the Cartesian product  $A \times A$  of a set  $A$ . It is easy to see that three points  $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in A \times A$  are collinear if  $\frac{b_1 - a_1}{c_1 - a_1} = \frac{b_2 - a_2}{c_2 - a_2} := \lambda$ . Thus any  $\lambda$  generates a linear equation  $b + (\lambda - 1)a - \lambda c = 0$ ,  $a, b, c \in A$  and hence the number of collinear triples is connected with a system of linear equations of type (2), for more details, see section 7. Further applications can be found in this section.

The paper is organized as follows. In sections 2, 4 we give a list of definitions and results, which will be used further in the text. In section 3 we consider some examples of families of sets avoiding several equations and prove lower bound (4). Section 5 is devoted to the spectrum of a set. Here we prove in particular, that the spectrum has small multiplicative energy and contains a large subset with even smaller multiplicative energy. In the next section we obtain our main Theorem 28 which implies Theorem 1. Finally, section 7 contains further applications of the main result.

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## 2 Definitions

Let  $p$  be a prime number,  $\mathbb{F}_p$  be the finite field and denote by  $\mathbb{F}_p^*$  the set  $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ . The field  $\mathbb{F}_p$  is the main subject of our paper but let us consider a slightly general context which we will use sometimes.

Let  $\mathbf{G}$  be an abelian group. If  $\mathbf{G}$  is finite, then denote by  $N$  the cardinality of  $\mathbf{G}$ . It is well-known [8] that the dual group  $\widehat{\mathbf{G}}$  is isomorphic to  $\mathbf{G}$  in this case. Let  $f$  be a function from

$\mathbf{G}$  to  $\mathbb{C}$ . We denote the Fourier transform of  $f$  by  $\widehat{f}$ ,

$$\widehat{f}(\xi) = \sum_{x \in \mathbf{G}} f(x) e(-\xi \cdot x), \quad (5)$$

where  $e(x) = e^{2\pi i x}$  and  $\xi$  is a homomorphism from  $\widehat{\mathbf{G}}$  to  $\mathbb{R}/\mathbb{Z}$  acting as  $\xi : x \rightarrow \xi \cdot x$ . We rely on the following basic identities

$$\sum_{x \in \mathbf{G}} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2, \quad (6)$$

$$\sum_{y \in \mathbf{G}} \left| \sum_{x \in \mathbf{G}} f(x) g(y - x) \right|^2 = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2 |\widehat{g}(\xi)|^2, \quad (7)$$

and

$$f(x) = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} \widehat{f}(\xi) e(\xi \cdot x). \quad (8)$$

If

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y) g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y) g(y + x),$$

then

$$\widehat{f * g} = \widehat{f} \widehat{g} \quad \text{and} \quad \widehat{f \circ g} = \widehat{f}^c \widehat{g} = \overline{\widehat{f}} \widehat{g}, \quad (9)$$

where for a function  $f : \mathbf{G} \rightarrow \mathbb{C}$  we put  $f^c(x) := f(-x)$ . Clearly,  $(f * g)(x) = (g * f)(x)$  and  $(f \circ g)(x) = (g \circ f)(-x)$ ,  $x \in \mathbf{G}$ . The  $k$ -fold convolution,  $k \in \mathbb{N}$  we denote by  $*_k$ , so  $*_k := (*_{k-1})$ . In the same way we use multiplicative convolution of two functions  $f, g : \mathbb{F}_p \rightarrow \mathbb{C}$  which we denote as

$$(f \otimes g)(x) := \sum_{y \in \mathbb{F}_p^*} f(y) g(xy^{-1}).$$

Write for any function  $f : \mathbf{G} \rightarrow \mathbb{C}$

$$\|f\|'_\infty := \max_{x \neq 0} |f(x)|.$$

We use in our paper the same letter to denote a set  $S \subseteq \mathbf{G}$  and its characteristic function  $S : \mathbf{G} \rightarrow \{0, 1\}$ . Write  $\mathbf{E}^+(A, B)$  for the *additive energy* of two sets  $A, B \subseteq \mathbf{G}$  (see, e.g., [22]), that is,

$$\mathbf{E}^+(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

If  $A = B$  we simply write  $\mathbf{E}^+(A)$  instead of  $\mathbf{E}^+(A, A)$ . In the same way one can define the *multiplicative energy* of two sets  $A, B \subseteq \mathbb{F}_p$  as

$$\mathbf{E}^\times(A, B) = |\{a_1 b_1 = a_2 b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

Sometimes we write  $\mathbf{E}(A, B)$  if we do not specialise the energy. Further clearly,

$$\mathbf{E}^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x) (B \circ B)(x). \quad (10)$$

and by (7),

$$\mathbf{E}^+(A, B) = \frac{1}{N} \sum_{\xi} |\widehat{A}(\xi)|^2 |\widehat{B}(\xi)|^2. \quad (11)$$

Also put

$$\mathbf{E}_*^+(A) = \frac{1}{N} \sum_{\xi \neq 0} |\widehat{A}(\xi)|^4 = \mathbf{E}^+(A) - \frac{|A|^4}{N}.$$

Let

$$\mathbf{T}_k^+(A) := |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k : a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}| = \frac{1}{N} \sum_{\xi} |\widehat{A}(\xi)|^{2k}.$$

Also let

$$\sigma_k^+(A) := (A *_k A)(0) = |\{a_1 + \dots + a_k = 0 : a_1, \dots, a_k \in A\}|.$$

Notice that for a symmetric set  $A$ , that is,  $A = -A$  one has  $\sigma_2(A) = |A|$  and  $\sigma_{2k}^+(A) = \mathbf{T}_k^+(A)$ . Having a set  $P \subseteq A - A$  we write  $\sigma_P^+(A) := \sum_{x \in P} (A \circ A)(x)$  and  $\mathbf{E}_P^+(A) := \sum_{x \in P} (A \circ A)^2(x)$ .

Given two sets  $Q, R \subseteq \mathbf{G}$  and a real number  $t \geq 1$ , we define

$$\text{Sym}_t^+(Q, R) := \{x : |Q \cap (x - R)| \geq t\}$$

and similar  $\text{Sym}_t^\times(Q, R)$ .

For a positive integer  $n$ , we set  $[n] = \{1, \dots, n\}$ . All logarithms are to base 2. Signs  $\ll$  and  $\gg$  are the usual Vinogradov's symbols, that is,  $a \ll b$  iff  $a = O(b)$ . We will write  $a \lesssim b$  or  $b \gtrsim a$  if  $a = O(b \cdot \log^c |A|)$ , where  $A$  is a fixed set and  $c > 0$  is an absolute constant. Notation  $a \sim b$  means  $a \lesssim b$  and, simultaneously,  $b \lesssim a$ .

### 3 Examples of sets avoiding several linear equations

First of all, let us recall the definitions. Let  $\mathcal{E}$  be a finite family of equations of the form

$$a_j x + b_j y + c_j z = d_j, \quad (12)$$

where all  $a_j, b_j, c_j$  are nonzero and such that any two triples  $(a_j, b_j, c_j), (a'_j, b'_j, c'_j)$  corresponding some equations from  $\mathcal{E}$  are not proportional. In other words, we consider triples  $(a_j, b_j, c_j)$  from  $\mathbb{F}_p^* \times \mathbb{F}_p^* \times \mathbb{F}_p^*$  up to an equivalence relation  $\sim$ , namely,  $(a, b, c) \sim (a', b', c')$  iff for some nonzero  $\lambda$  the following holds  $a' = \lambda a$ ,  $b' = \lambda b$ , and  $c' = \lambda c$ . Thus the family  $\mathcal{E}$  corresponds to a subset of two-dimensional projective plane. We denote this set as  $S(\mathcal{E})$ . We write  $|\mathcal{E}|$  for the cardinality of  $S(\mathcal{E})$ . Also notice that we do not require  $d_j = 0$  or  $a_j + b_j + c_j = 0$  and hence so-called non-affine equations (see [11, 12]) are considered by us as well. We say that a set  $A \subseteq \mathbb{F}_p$  is *avoiding family*  $\mathcal{E}$  if there is no  $j \in [|\mathcal{E}|]$  and  $x, y, z \in A$  such that  $a_j x + b_j y + c_j z = d_j$ . In other words, the set  $A$  does not satisfy *all* equations from  $\mathcal{E}$ . Sometimes a little bit more general setting is required. Let  $A_1, A_2, A_3 \subseteq \mathbb{F}_p$  be three sets. We say that the triple  $(A_1, A_2, A_3)$  *avoids family*  $\mathcal{E}$  if for any  $j \in [|\mathcal{E}|]$  and all  $x \in A_1, y \in A_2, z \in A_3$  we have  $a_j x + b_j y + c_j z \neq d_j$ .

Of course the size of a set  $A$  avoiding equations (12) depends on the geometry of the set  $\mathcal{E}$  or, equivalently, on the set  $S(\mathcal{E})$ . We consider several rather rough characteristics of the set  $\mathcal{E}$  and study them.

**Definition 2** By  $\mathcal{T}(\mathcal{E})$  denote the size of the maximal subset in the intersection of  $\mathcal{E}$  with one of three planes  $\{x = 1\}, \{y = 1\}, \{z = 1\}$  with the property that all non-fixed coordinates in the intersection are different.

Thus  $\mathcal{T}(\mathcal{E}) \leq |\mathcal{E}|$  and the bound is attained if, say,  $S(\mathcal{E}) = \{1\} \times \{(e, e) : e \in [|\mathcal{E}|]\}$ . Now let us obtain a lower bound for the quantity  $\mathcal{T}(\mathcal{E})$ .

**Lemma 3** We have  $\mathcal{T}(\mathcal{E}) \geq |\mathcal{E}|^{1/2}$ .

**Proof.** Put  $s = |S(\mathcal{E})| = |\mathcal{E}|$ ,  $t = \mathcal{T}(\mathcal{E})$ . Take a maximal subset  $J$  of  $[|\mathcal{E}|]$  such that  $(x_j, y_j, 1) \in \mathcal{E}$  and all elements  $x_j$  as well as all elements  $y_j$  are different. Clearly,  $t \geq |J|$ . Put  $R = \{(x_j, y_j)\}_{j \in J}$  and  $S = \{(x_j, y_j)\}_{j \in [s]}$ . By the maximality of  $R$  we see that any point of  $S$  has either the same abscissa or the same ordinate with a point from  $R$ . Thus one can split  $S \setminus R$  into two sets  $S_1, S_2$  and  $R$  into sets  $R_1, R_2$  such that any point from  $S_1, S_2$  shares common abscissa or ordinate (or both) with some point from  $R_1, R_2$ , respectively. Let us split points from  $S \setminus R$  having common abscissa and ordinate with some points from  $R$  in an arbitrary way. Let  $r_1 = |R_1|$ ,  $r_2 = |R_2|$ ,  $s_1 = |S_1|$ ,  $s_2 = |S_2|$ . Then, clearly,  $r_1 + r_2 = |J|$  and  $s_1 + s_2 + r_1 + r_2 = s$ . Suppose that  $r_1, r_2 > 0$ . By average arguments there is some point  $x_0$  such that the set  $\{(x_0, y, 1) : (x_0, y) \in S_1\}$  has size at least  $s_1/r_1$ . Similarly, there is some point  $y_0$  such that the set  $\{(x, y_0, 1) : (x, y_0) \in S_2\}$  has size at least  $s_2/r_2$ . Without losing of generality suppose that  $\frac{s_2}{r_2} \leq \frac{s_1}{r_1}$ . By a well-known property of the median, we have

$$\frac{s_2}{r_2} \leq \frac{s_1 + s_2}{r_1 + r_2} = \frac{s}{|J|} - 1 \leq \frac{s_1}{r_1}.$$

Hence there is a set  $Q$  of the form  $Q = \{(x_0, q, 1) \in S\}$  of size  $|Q| \geq s/|J| - 1 + 1 = s/|J|$  (we add in  $Q$  a point from  $R_1$ ). If  $r_1$  or  $r_2$  vanishes then it is easy to see that the existence of such  $Q$  follows similarly and even simpler. Finally, the points  $(x_0, q, 1)$  from  $Q$  are equivalent to  $|Q|$  points of the form  $(x_0 q^{-1}, 1, q^{-1})$ , having different coordinates in the plane  $\{y = 1\}$ . Thus,  $t \geq s/|J|$ . Obviously,  $\max_{|J|} \{|J|, s/|J|\} \geq s^{1/2}$  and hence  $t \geq s^{1/2}$ . This completes the proof.  $\square$

**Remark 4** Let  $\Gamma$  be a subgroup of  $\mathbb{F}_p^*$  and  $S(\mathcal{E}) = (\Gamma \times \Gamma \times \Gamma) / \sim$ . Then in view of  $\Gamma/\Gamma = \Gamma$ , we have  $|\mathcal{E}| = |\Gamma|^2$  and it is easy to see that  $\mathcal{T}(\mathcal{E}) = |\Gamma| = |\mathcal{E}|^{1/2}$ . It follows that the bound of Lemma 3 is tight.

Let us consider another characteristic of the set  $S(\mathcal{E})$ .

**Definition 5** Take the intersection of  $\mathcal{E}$  with any of three planes  $\{x = 1\}, \{y = 1\}, \{z = 1\}$ , say with  $\{z = 1\}$ . We obtain points  $(a_j, b_j, 1) \in \mathcal{E}$ . Then by  $\mathcal{T}_*(\mathcal{E})$  denote the size of a maximal subset  $J$  of  $[|\mathcal{E}|]$  such that for any  $j \in J$  either  $a_j \neq a_i$  or  $b_j \neq b_i$  or  $a_j b_j^{-1} \neq a_i b_i^{-1}$  for all  $i \in J$ ,  $i \neq j$ . In other words, each point  $(a_j, b_j)$  has either a unique abscissa or ordinate or its ratio.

Thus  $\mathcal{T}_*(\mathcal{E}) \leq |\mathcal{E}|$  and the bound it attained if, say,  $S(\mathcal{E}) = \{1\} \times \{1\} \times [|\mathcal{E}|]$ . Now we obtain a lower bound for the quantity  $\mathcal{T}_*(\mathcal{E})$ .

**Lemma 6** *We have  $\mathcal{T}_*(\mathcal{E}) \geq 2|\mathcal{E}|^{1/2} - 1$ .*

**Proof.** Consider the intersection of  $\mathcal{E}$  with any of three planes  $\{x = 1\}, \{y = 1\}, \{z = 1\}$ , say with  $\{z = 1\}$  and put  $s = |S(\mathcal{E})| = |\mathcal{E} \cap \{z = 1\}|$ . So we can think about  $S(\mathcal{E})$  as a subset of  $\mathbb{F}_p^* \times \mathbb{F}_p^*$ . Take minimal sets  $A, B \subseteq \mathbb{F}_p^*$  such that  $S(\mathcal{E}) \subseteq A \times B$ . Put  $a = |A|$  and  $b = |B|$ . Then  $s \leq ab$ . Clearly, there is  $x_*$  such that  $|\{x = x_*\} \cap S(\mathcal{E})| \geq s/a$  and, similarly, there exists  $y_*$  with  $|\{y = y_*\} \cap S(\mathcal{E})| \geq s/b$ . Put  $V = \{x = x_*\} \cap S(\mathcal{E})$  and  $H = \{y = y_*\} \cap S(\mathcal{E})$ . If  $V \cap H = \emptyset$  then each point in  $V \cup H$  has either a unique abscissa or a unique ordinate. Now suppose that  $V \cap H = (x_*, y_*)$ . Then it is easy to see that the point  $(x_*, y_*)$  has a unique ratio  $x_*/y_*$  differs from ratios of points in  $V \cup H$ . Thus any point in  $V \cup H$  has either a unique abscissa or ordinate or its ratio. It gives us  $\mathcal{T}_*(\mathcal{E}) \geq s/a + s/b - 1$ . Optimizing the expression  $s/a + s/b - 1$  over  $a, b$  subject to  $s \leq ab$ , we get  $\mathcal{T}_*(\mathcal{E}) \geq 2|\mathcal{E}|^{1/2} - 1$  as required.  $\square$

**Remark 7** *Let  $\Gamma$  be a subgroup of  $\mathbb{F}_p^*$  and  $S(\mathcal{E}) = (\Gamma \times \Gamma \times \Gamma)/\sim$ . Then we have  $\Gamma/\Gamma = \Gamma$  and hence  $\mathcal{T}_*(\mathcal{E}) \leq 3|\Gamma| = 3|\mathcal{E}|^{1/2}$  (actually one can show that*

$$\mathcal{T}_*(\mathcal{E}) \geq \max_{X, Y \subseteq \Gamma} (|X/Y| + 2|\Gamma| - |X| - |Y| - 1)$$

*and thus a lower bound for  $\mathcal{T}_*(\mathcal{E})$  is  $(3 - o(1))|\Gamma|$ . We do not need this fact.) It follows that the bound of Lemma 6 is tight up to constants.*

We say that  $S(\mathcal{E})$  forms the Cartesian product if  $S(\mathcal{E})$  is equivalent to the Cartesian product  $A \times B \times \{1\}$  or  $A \times \{1\} \times B$  or  $\{1\} \times A \times B$  in two-dimensional projective plane. With some abuse of the notation we write sometimes  $S(\mathcal{E}) = A \times B$  in this case. Notice that always

$$\mathcal{T}_*(\mathcal{E}) \geq |\{x/y : (x, y, 1) \in S(\mathcal{E})\}|. \quad (13)$$

In particular, if  $S(\mathcal{E}) = A \times B$ , then  $\mathcal{T}_*(\mathcal{E}) \geq |A/B|$  but it is easy to see that  $\mathcal{T}(\mathcal{E}) = \max\{|A|, |B|\}$ . Thus the quantities  $\mathcal{T}(\mathcal{E}), \mathcal{T}_*(\mathcal{E})$  are incomparable in general although we have a trivial inequality  $\mathcal{T}(\mathcal{E}) \leq \mathcal{T}_*(\mathcal{E})$  of course.

Now we consider several examples of concrete systems of equations (12).

**Example 8** *Let  $\Gamma \subseteq \mathbb{F}_p^*$  be a multiplicative subgroup. We are interested in basis properties of  $\Gamma$ , that is, in a question when  $\Gamma + \Gamma$  contains  $\mathbb{F}_p^*$ . If  $\Gamma + \Gamma$  does not contain  $\mathbb{F}_p^*$ , then it is easy to see that for some nonzero  $\xi$  one has  $(\Gamma + \Gamma) \cap \xi\Gamma = \emptyset$ . It means that, taking any  $a, b \in \Gamma$  the equation  $ax + by - \xi z = 0$  has no solutions in  $\Gamma$ . Thus  $S(\mathcal{E})$  is the Cartesian product in this case.*

*Similarly, one can consider a slightly general situation and study sets  $A$  with  $(A + A) \cap \Gamma = \emptyset$ , where  $A$  is not necessary  $\Gamma$ -invariant. Here the equation  $\gamma x + \gamma y - z = 0$ ,  $x, y \in A$ ,  $z \in \Gamma$  has no solutions for any  $\gamma \in \Gamma$  and thus  $S(\mathcal{E}) = \{(\gamma, \gamma, -1) : \gamma \in \Gamma\}$  for corresponding triple  $(A, A, \Gamma)$ .*

Proposition below shows that one cannot replace the constant  $\kappa$  in Theorem 1 and  $\kappa_1, \kappa_2, \kappa_3$  in Theorem 28 below by something greater than  $1/2$ .

**Proposition 9** *For any  $k \geq 1$  there is a system  $\mathcal{E}$  with  $\mathcal{T}_*(\mathcal{E}) \gg |\mathcal{E}| \geq k$  such that for all sufficiently large  $p \geq p(k)$  there exists  $A \subseteq \mathbb{F}_p$  avoiding the family  $\mathcal{E}$  with*

$$|A| \gg \frac{p}{|\mathcal{E}|^{1/2}}. \quad (14)$$

**Proof.** Let  $2 \leq q < \sqrt{p}$  be an even parameter and

$$A := \{1 \leq x < p/q : x \equiv 1 \pmod{2}\}.$$

We have  $|A| \gg p/q$ . Put  $q' = q/2$  and  $S = \{(i, j) \in [q'] : i, j \equiv 0 \pmod{2}\}$ . Clearly,  $q^2 \geq |S| \gg q^2$ , so taking  $p$  and hence  $q$  sufficiently large, we get  $|S| \geq k$ . Finally, let  $\mathcal{E}$  be the Cartesian product with  $\{z = 1\}$  and let  $S(\mathcal{E}) = -S$ , so  $\mathcal{E} \sim (-S) \times \{1\}$ . Also let us put  $d_j = 0$  in (12). Once again, we have proved that  $|\mathcal{E}| \geq k$  already.

First of all, let us prove that  $\mathcal{T}_*(\mathcal{E}) \gtrsim |\mathcal{E}| \gg q^2$ . Consider the square-free numbers  $F_0$  from  $[q']$ , notice that  $|F_0| \gg q' \gg q$  and easily check the identity  $|F_0/F_0| = |F_0|^2 \gg q^2$ . After that we take the set  $S(\mathcal{E}') := \{(i, j) \in F_0^2 : (i, j) \in S\} \subseteq S(\mathcal{E})$  and we get by (13) and the fact  $|F_0/F_0| = |F_0|^2$  that  $\mathcal{T}_*(\mathcal{E}) \geq \mathcal{T}_*(\mathcal{E}') \gg q^2$ .

Secondly, we need to check that  $A$  avoiding the family  $\mathcal{E}$ . If not, then there are  $x, y, z \in A$  such that  $ix + jy \equiv z \pmod{p}$ . We have  $|ix + jy - z| < 2q' \cdot p/q = p$  and hence  $ix + jy = z$ . Thus, from  $i, j \equiv 0 \pmod{2}$ , it follows that  $z = ix + jy \equiv 0 \pmod{2}$  but by the definition of the set  $A$ , we know that  $z \equiv 1 \pmod{2}$  which is a contradiction. This completes the proof.  $\square$

**Remark 10** *In [23], [2] it was constructed a multiplicative subgroup  $\Gamma \subseteq \mathbb{F}_q \setminus \{0\}$  with  $|\Gamma| \gg q^{2/3}$  and having no solutions of the equation  $x + y + z = 0$ . In view of Example 8 it gives another system  $\mathcal{E}$  such that (14) holds in a general field  $\mathbb{F}_q$ . Here  $\mathcal{T}(\mathcal{E}) \sim \mathcal{T}_*(\mathcal{E}) \sim |\Gamma| = |\mathcal{E}|^{1/2}$ , see Remarks 4, 7.*

*If we consider the case of prime  $p$ , then a multiplicative subgroup  $\Gamma \subseteq \mathbb{F}_p^*$  without solutions of the equation  $x + y = z$  was constructed with the constraint  $|\Gamma| \gg p^{1/3}$  only, see [2]. Conjecturally, the right bound here is  $|\Gamma| \geq p^{1/2-o(1)}$  or even  $p^{2/3-o(1)}$ , see [2, Section 4].*

We finish this section considering another family  $\mathcal{E}$ .

**Example 11** *Take the family of equations*

$$x + y + \lambda z = 0, \quad (15)$$

*where  $\lambda \in \Lambda$  and  $\Lambda \subseteq \mathbb{F}_p^*$  is a set. We have  $t := |\Lambda| = \mathcal{T}(\mathcal{E}) = \mathcal{T}_*(\mathcal{E}) = |\mathcal{E}|$  for this family. By the main result from [15] the equation*

$$a + b = cd, \quad a \in A, b \in B, c \in C, d \in D$$

has a solution if  $|A||B||C||D| \gg p^3$ . In other words, if  $A$  avoids all equations (15), then  $|A| \ll p/t^{1/3}$ . The same bound holds in the case of the Cartesian product (to see this just fix a variable, say,  $b$  in the correspondent equation  $a + bc + de = 0$ .) An improvement of  $1/3$  would imply a considerable progress in the area (in particular, for basis properties of multiplicative subgroups) and seems unattainable at the moment.

Further examples of families  $\mathcal{E}$  can be found in the last section 7.

## 4 Preliminaries

Let us begin with a simple lemma about the triangle inequality for restricted energies.

**Lemma 12** *For any four sets  $A, B, X, Y \subseteq \mathbf{G}$  one has*

$$\mathbf{E}_A(X \sqcup Y, B)^{1/2} \leq \mathbf{E}_A(X, B)^{1/2} + \mathbf{E}_A(Y, B)^{1/2}.$$

*Proof.* We have

$$\begin{aligned} \mathbf{E}_A(X \sqcup Y, B) &= \sum_{a \in A} ((X \circ B)(a) + (Y \circ B)(a))^2 = \\ &= \mathbf{E}_A(X, B) + \mathbf{E}_A(Y, B) + 2 \sum_{a \in A} (X \circ B)(a)(Y \circ B)(a). \end{aligned}$$

By the Cauchy–Schwarz inequality, we get

$$\mathbf{E}_A(X + Y, B) \leq \mathbf{E}_A(X, B) + \mathbf{E}_A(Y, B) + 2\mathbf{E}_A^{1/2}(X, B)\mathbf{E}_A^{1/2}(Y, B) = (\mathbf{E}_A^{1/2}(X, B) + \mathbf{E}_A^{1/2}(Y, B))^2$$

as required.  $\square$

Now we need in some sum–product results. In [1] it was shown that the main theorem from [9] implies the following weaker version of the Szemerédi–Trotter Theorem for a general field  $\mathbb{F}$ .

**Theorem 13** *Let  $A, B \subset \mathbb{F}$  be two sets, with  $|B| \leq |A| \leq p^{2/3}$  in positive characteristic. The number of incidences between the point set  $A \times B$  and any set of  $m$  lines in  $\mathbb{F}^2$  is*

$$O((|A||B|)^{3/4}m^{2/3} + m + |A||B|).$$

In [1] (see also [7]) it was obtained another particular sum–product result which we will use in the next sections. For more general context consult with paper [1].

**Theorem 14** *Suppose that  $A, B, C \subseteq \mathbb{F}_p$  are sets with  $|A||B||C| = O(p^2)$ . Then*

$$\begin{aligned} &|\{(a_1, a_2, b_1, b_2, c_1, c_2) \in A^2 \times B^2 \times C^2 : a_1(b_1 + c_1) = a_2(b_2 + c_2)\}| \ll \\ &\ll (|A||B||C|)^{\frac{3}{2}} + |A||B||C| \max\{|A|, |B|, |C|\}. \end{aligned}$$



Let us obtain a consequence of Theorem 13 in the spirit of paper [5].

**Lemma 15** *Let  $A \subset \mathbb{F}_p$  be a set such that  $A \subseteq \text{Sym}_t^+(P, Q)$ , where  $P, Q \subseteq \mathbb{F}_p$  are two another sets with  $|P|^2|Q|^{5/4} \geq t^2|A|^{3/4}$  and  $|Q| < p^{2/3}$ . Then for any set  $B$ ,  $|B| < p^{2/3}$  one has*

$$|\{s : |A \cap Bs| \geq \tau\}| \ll \frac{|P|^2|Q|^{9/4}|B|^{9/4}}{t^3\tau^3}. \quad (16)$$

*Proof.* Let  $S_\tau$  be the set in the left-hand side of (16). We have

$$\tau|S_\tau| \leq \sum_{s \in S_\tau} |A \cap Bs| = |\{a = bs : a \in A, b \in B, s \in S_\tau\}| := \sigma.$$

Because  $A \subseteq \text{Sym}_t^+(P, Q)$ , we obtain the following upper bound for the number of solutions  $\sigma$

$$\sigma \leq t^{-1} |\{p + q = sb : p \in P, q \in Q, b \in B, s \in S_\tau\}|. \quad (17)$$

First of all, let us prove a trivial estimate for the size of  $S_\tau$ . Namely, dropping the condition  $s \in S_\tau$  in (17), we get

$$\tau|S_\tau|t \leq |P||Q||B|$$

and hence inequality (16) should be checked in the range

$$t^2\tau^2 \gg |P||Q|^{5/4}|B|^{5/4} \quad (18)$$

only because otherwise

$$|S_\tau| \leq \frac{|P||Q||B|}{t\tau} \ll \frac{|P|^2|Q|^{9/4}|B|^{9/4}}{t^3\tau^3}.$$

Let us notice one consequence of (18). Using inequality (18) again as well as trivial bounds  $\tau \leq |B|$  and  $t \leq |Q|$ , we have

$$|P| \ll |Q|^{3/4}|B|^{3/4}. \quad (19)$$

Further consider the family  $\mathcal{L}$  of  $|P||S_\tau|$  lines  $l_{p,s} = \{(x, y) : p + y = sx\}$ ,  $p \in P$ ,  $s \in S_\tau$  and the family of points  $\mathcal{P} = B \times Q$ . By our assumptions, we have  $|B|, |Q| < p^{2/3}$ . Applying Theorem 13 to the pair  $(\mathcal{P}, \mathcal{L})$ , we get

$$\sigma \leq t^{-1}\mathcal{I}(\mathcal{P}, \mathcal{L}) \ll t^{-1} \left( |\mathcal{P}|^{3/4}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}| \right). \quad (20)$$

If the first term in (20) dominates, then we obtain (16). Now suppose that required bound (16) does not hold. Then if the second term in (20) is the largest one, we obtain

$$\frac{|P|^2|Q|^{9/4}|B|^{9/4}}{t^2\tau^2} \ll t\tau|S_\tau| \ll |P| = |Q||B|.$$

But, clearly,  $t \leq \min\{|P|, |Q|\}$  and  $\tau \leq \min\{|A|, |B|\}$ , thus

$$|P|^2|Q|^{5/4} \ll t^2|A|^{3/4}$$

and we arrive to a contradiction with our assumption  $|P|^2|Q|^{5/4} \geq t^2|A|^{3/4}$  (actually, we need in  $|P|^2|Q|^{5/4} \gg t^2|A|^{3/4}$  but we can assume just  $|P|^2|Q|^{5/4} \geq t^2|A|^{3/4}$  increasing the constants in our main bound (16)). Finally, we need to consider the case when the third term in (20) dominates. In this situation

$$t\tau|S_\tau| \ll |\mathcal{L}| = |S_\tau||P|$$

and hence in view of (18)

$$|P||Q|^{5/4}|B|^{5/4} \ll |P|^2. \quad (21)$$

Recalling (19), we obtain

$$|Q|^{5/4}|B|^{5/4} \ll |Q|^{3/4}|B|^{3/4}.$$

But this is a contradiction if  $Q$  or  $B$  is large enough. This completes the proof.  $\square$

By a simple summation we get an immediate consequence of the last lemma.

**Corollary 16** *Let  $A \subset \mathbb{F}_p$  be a set such that  $A \subseteq \text{Sym}_t^+(P, Q)$ , where  $P, Q \subseteq \mathbb{F}_p$  are two another sets with  $|P|^2|Q|^{5/4} \geq t^2|A|^{3/4}$  and  $|Q| < p^{2/3}$ . Then for any sets  $X, Y$ ,  $|Y| < p^{2/3}$  one has*

$$E_X^\times(A, Y) \ll \frac{|P|^{4/3}|Q|^{3/2}|Y|^{3/2}|X|^{1/3}}{t^2}.$$

## 5 The multiplicative energy of the spectrum

Now recall the notion of the spectrum  $\text{Spec}_\varepsilon(A)$  of a set  $A$  and formulate the required result about the structure of  $\text{Spec}_\varepsilon(A)$ . Let  $A \subseteq \mathbb{F}_p$  be a set, and  $\varepsilon \in (0, 1]$  be a real number. Define

$$\text{Spec}_\varepsilon(A) = \{r \in \mathbb{F}_p : |\widehat{A}(r)| \geq \varepsilon|A|\}.$$

Clearly,  $0 \in \text{Spec}_\varepsilon(A)$ , and  $\text{Spec}_\varepsilon(A) = -\text{Spec}_\varepsilon(A)$ . In this section we denote by  $\delta$  the density of our set  $A$ , that is,  $\delta = |A|/p$ . From Parseval identity (6), we have a simple upper bound for the size of the spectrum, namely,

$$|\text{Spec}_\varepsilon(A)| \leq \frac{p}{|A|\varepsilon^2} = \frac{1}{\delta\varepsilon^2}. \quad (22)$$

We need in a result from [17] (tight bounds are contained in paper [18]) which shows that the spectrum has a rich additive structure.

**Lemma 17** *Let  $A \subset \mathbb{F}_p$  be a set,  $k \geq 2$  be an integer, and  $\varepsilon \in (0, 1]$  be a real number. Then for any  $B \subseteq \text{Spec}_\varepsilon(A)$  one has*

$$\mathsf{T}_k^+(B) \geq \varepsilon^{2k}|B|^{2k} \cdot |A|/p. \quad (23)$$

We need in Proposition 16 from [10] and a combinatorial lemma which is contained in the proof of this proposition. We give the proof of this lemma for completeness.

**Lemma 18** *Let  $A \subseteq \mathbf{G}$  be a set,  $P \subseteq A - A$ ,  $P = -P$ . Then there is  $A_* \subseteq A$  and a number  $q$ ,  $q \lesssim |A_*|$  such that for any  $x \in A_*$  one has  $(A * P)(x) \geq q$ , and  $\sigma_P(A) \sim |A_*|q$ .*

*Proof.* We have

$$\sigma := \sigma_P(A) = \sum_{x \in P} (A \circ A)(x) = \sum_{x \in A} (A * P)(x).$$

Using the pigeonhole principle, we find  $A' \subseteq A$  such that  $(A * P)(x)$  differ by a multiplicative factor of at most twice on  $A'$  and  $\sigma \sim q'|A'|$ , where  $q' = \min_{x \in A'} (A * P)(x)$ . If  $q' \leq |A'|$ , then put  $A_* = A'$ ,  $q = q'$  and we are done. Suppose not. By assumption  $P = -P$ , and thus we get

$$\sigma \lesssim \sum_{x \in A'} (A * P)(x) = \sum_{x \in A} (A' * P)(x).$$

Then applying the pigeonhole principle one more time, we find  $A'' \subseteq A$  such that  $(A' * P)(x)$  differ by a multiplicative factor of at most twice on  $A''$  and  $\sigma \sim q''|A''|$ , where  $q'' = \min_{x \in A''} (A' * P)(x)$ . Using the inequality  $q' > |A'|$  and a trivial bound  $q'' \leq |A'|$ , we obtain

$$|A''||A'| \geq |A''|q'' \gtrsim \sigma \geq |A'|q' > q''|A'|$$

and hence  $q'' \lesssim |A''|$ . After that we put  $A_* = A''$ ,  $q = q''$ . This completes the proof.  $\square$

Recall a result from [10], see Proposition 16 from here.

**Proposition 19** *Let  $S \subseteq \mathbb{F}_p$  be a set,  $|S|^6 \lesssim p^2 \mathbf{E}^\times(S)$ . Then there is a set  $S_1 \subseteq S$ ,  $|S_1|^2 \gtrsim \mathbf{E}^\times(S)/|S|$  and*

$$\mathbf{E}^+(S_1)^2 \mathbf{E}^\times(S)^3 \lesssim |S_1|^{11} |S|^3. \quad (24)$$

*The same result holds if one replace  $+$  onto  $\times$  and vice versa.*

Now we are ready to prove that any (large) subset of the spectrum is always has small multiplicative energy. This is one of the main results of this section.

**Theorem 20** *Let  $A \subset \mathbb{F}_p$  be a set, and  $\varepsilon \in (0, 1]$  be a real number. Then for any  $B \subseteq \text{Spec}_\varepsilon(A)$ ,  $|B| < \delta^{-1/6} \varepsilon^{-2/3} \sqrt{p}$ , one has*

$$\mathbf{E}^\times(B) \lesssim |B|^2 \cdot \delta^{-2/3} \varepsilon^{-8/3}. \quad (25)$$

*Proof.* If  $\mathbf{E}^\times(B) \lesssim |B|^2 \cdot \delta^{-2/3} \varepsilon^{-8/3}$ , then it is nothing to prove. Otherwise, in view of our assumption, we have  $|B|^6 \lesssim p^2 \mathbf{E}^\times(B)$ . Applying Proposition 19 with  $S = B$ , we find  $B_1 \subseteq B$  such that  $|B_1|^2 \gtrsim \mathbf{E}^\times(B)/|B|$  and

$$\mathbf{E}^+(B_1)^2 \mathbf{E}^\times(B)^3 \lesssim |B_1|^{11} |B|^3.$$

Further, using Lemma 17 for  $B = B_1$  and  $k = 2$ , we have  $\mathbf{E}^+(B_1) \geq \delta \varepsilon^4 |B_1|^4$ . Thus

$$\mathbf{E}^\times(B)^3 \lesssim |B_1|^3 |B|^3 \delta^{-2} \varepsilon^{-8} \leq |B|^6 \delta^{-2} \varepsilon^{-8}$$

as required.  $\square$

**Example 21** Let  $\varepsilon \gg 1$ ,  $B = \text{Spec}_\varepsilon(A)$ ,  $|A| \gg p^{2/5}$  and the size of  $B$  is comparable with the upper bound which is given by (22), namely,  $|B| \gg \delta^{-1}$ . Then  $E^\times(B) \lesssim |B|^{8/3}$ . It means that we have a non-trivial estimate for the multiplicative energy of the spectrum in this case.

**Remark 22** The proof of Theorem 20 shows that a similar statement about the energy holds for wider class of so-called connected sets, that is, sets  $S$  such that for any  $S' \subseteq S$ ,  $|S'| \gg |S|$  one has  $E(S') \gg E(S)$ , see the rigorous definition in [19], say.

It is possible to increase the size of the set  $S_1$  in Proposition 19 decreasing the upper estimate for the product of energies in (24). Our arguments mimic the proof of Corollary 22 from [5].

**Corollary 23** Let  $S \subseteq \mathbb{F}_p$  be a set,  $|S|^6 \lesssim p^2 E^\times(S)$ . Then there is a set  $S' \subseteq S$ ,  $|S'|^3 \gtrsim E^\times(S)$  and

$$E^+(S')^2 E^\times(S)^3 \lesssim |S|^{14}. \quad (26)$$

The same result holds if one replace  $+$  onto  $\times$  and vice versa.

**Proof.** Our arguments is a sort of an algorithm. We construct a decreasing sequence of sets  $U_1 = S \supseteq U_2 \supseteq \dots \supseteq U_k$  and an increasing sequence of sets  $V_0 = \emptyset \subseteq V_1 \subseteq \dots \subseteq V_{k-1} \subseteq S$  such that for any  $j = 1, 2, \dots, k$  the sets  $U_j$  and  $V_{j-1}$  are disjoint and moreover  $S = U_j \sqcup V_{j-1}$ . If at some step  $j$  we have  $|V_j| > (E^\times(S))^{1/3}/2$ , then we stop our algorithm putting  $S' = V_j$  and  $k = j - 1$ . In the opposite situation we have  $|V_j| \leq (E^\times(S))^{1/3}/2$ . Applying Proposition 19 to the set  $U_j$ , we find the subset  $Y_j$  of  $U_j$  such that  $|Y_j|^2 \gtrsim E^\times(U_j)/|U_j|$  and such that

$$E^+(Y_j)^2 E^\times(U_j)^3 \lesssim |Y_j|^{11} |U_j|^3,$$

provided  $|U_j|^6 \lesssim p^2 E^\times(U_j)$ . Now notice that the inequality  $|V_j| \leq (E^\times(S))^{1/3}/2$  implies that  $E^\times(V_j) \leq |V_j|^3 \leq E^\times(S)/8$  and hence  $E^\times(U_j) \gg E^\times(S)$ . In particular,  $|Y_j|^2 \gtrsim E^\times(S)/|S|$  and, further

$$|U_j|^6 \leq |S|^6 \lesssim p^2 E^\times(S) \ll p^2 E^\times(U_j),$$

and thus the condition  $|U_j|^6 \lesssim p^2 E^\times(U_j)$  takes place. Hence

$$E^+(Y_j) \lesssim |Y_j|^{11/2} |S|^{3/2} E^\times(S)^{-3/2}.$$

After that we put  $U_{j+1} = U_j \setminus Y_j$ ,  $V_j = V_{j-1} \sqcup Y_j$  and repeat the procedure. Clearly, for each number  $k$ , we have  $V_k = \bigsqcup_{j=1}^k Y_j$  and it is easy to see that our algorithm must stop at some step  $k$ . Put  $S' = V_{k-1}$ . It is known that  $E^{1/4}(\cdot)$  is a norm, see, e.g., [22] or [5], say (this fact can be considered as an analog of Lemma 12 with  $A = \mathbb{F}_p$  as well), whence

$$\begin{aligned} E^+(S') &\leq \left( \sum_{j=1}^{k-1} E^+(Y_j)^{1/4} \right)^4 \lesssim |S|^{3/2} E^\times(S)^{-3/2} \left( \sum_{j=1}^{k-1} |Y_j|^{11/8} \right)^4 \leq \\ &\leq |S|^{3/2} E^\times(S)^{-3/2} |S|^{11/2} = |S|^7 E^\times(S)^{-3/2} \end{aligned}$$

as required.  $\square$

Using the corollary above we can prove that any subset of the spectrum has large subset with small multiplicative energy.

**Theorem 24** *Let  $A \subset \mathbb{F}_p$  be a set, and  $\varepsilon \in (0, 1]$  be a real number. Then for any  $B := \text{Spec}_\varepsilon(A)$ ,  $|B| < \delta^{1/2}\varepsilon^2 p$  there is  $B' \subseteq B$  such that  $|B'|^3 \gtrsim \delta\varepsilon^4|B|^4$  and*

$$\mathbf{E}^\times(B') \lesssim |B| \cdot \delta^{-3/2}\varepsilon^{-6}. \quad (27)$$

**Proof.** Suppose that  $|B|^6 \lesssim p^2 \mathbf{E}^+(B)$ . Then we use Corollary 23, reversing  $+$  onto  $\times$ . Thus, there is  $B' \subseteq B$ ,  $|B'|^3 \gtrsim \mathbf{E}^+(B)$  and

$$\mathbf{E}^\times(B')^2 \mathbf{E}^+(B)^3 \lesssim |B|^{14}. \quad (28)$$

Applying Lemma 17 with  $k = 2$ , we see that  $\mathbf{E}^+(B) \geq \delta\varepsilon^4|B|^4$ . It gives us, firstly,  $|B'|^3 \gtrsim \varepsilon\delta^4|B|^4$  and, secondly, from (28), it follows that

$$\mathbf{E}^\times(B')^2 \delta^3 \varepsilon^{12} |B|^{12} \lesssim |B|^{14}$$

as required. To check inequality  $|B|^6 \lesssim p^2 \mathbf{E}^+(B)$ , we recall that  $\mathbf{E}^+(B) \geq \delta\varepsilon^4|B|^4$ . This completes the proof.  $\square$

**Example 25** *Let  $\varepsilon \gg 1$ ,  $B = \text{Spec}_\varepsilon(A)$ ,  $|A| \gg p^{1/3}$  and the size of  $B$  is comparable with upper bound (22), namely,  $|B| \gg \delta^{-1}$ . Then by Theorem 24 we find a set  $B' \subseteq B$  such that  $\mathbf{E}^\times(B') \lesssim |B'|^{5/2}$  and  $|B'| \gtrsim |B|$ .*

Theorem 24 immediately implies

**Corollary 26** *Let  $A \subset \mathbb{F}_p$  be a set, and  $\varepsilon \in (0, 1]$  be a real number. Then for any  $B := \text{Spec}_\varepsilon(A)$ ,  $|B| < \delta^{1/2}\varepsilon^2 p$  there is  $\tilde{B} \subseteq B$  such that  $|\tilde{B}| \geq |B|/2$  and*

$$\mathbf{E}^\times(\tilde{B}) \lesssim \delta^{-17/6}\varepsilon^{-34/3}|B|^{-1/3} \leq \delta^{-5/2}\varepsilon^{-32/3}.$$

**Proof.** Applying Theorem 24 to the set  $B$ , we find  $B'_1 := B' \subseteq B$  such that (27) holds and  $|B'_1|^3 \gtrsim \delta\varepsilon^4|B|^4$ . Consider  $B \setminus B'_1$ . If  $|B \setminus B'_1| < |B|/2$ , then we are done. If not, then apply the same arguments to this set. An so on. At the end we have constructed a sequence of disjoint subsets of  $B$ , namely,  $B'_1, \dots, B'_k$  such that the set  $\tilde{B} := \bigsqcup_{j=1}^k B'_j$  has size at least  $|B|/2$ . Clearly,  $k \lesssim \delta^{-1/3}\varepsilon^{-4/3}|B|^{-1/3}$ . Because  $\mathbf{E}^{1/4}(\cdot)$  is a norm, we obtain in view of (27) and the Parseval identity that

$$\mathbf{E}^\times(\tilde{B}) \leq k^4 |B| \delta^{-3/2}\varepsilon^{-6} = \delta^{-17/6}\varepsilon^{-34/3}|B|^{-1/3} \leq \delta^{-5/2}\varepsilon^{-32/3}.$$

This completes the proof.  $\square$

As in Example 25 if  $\varepsilon \gg 1$ ,  $B = \text{Spec}_\varepsilon(A)$ ,  $|A| \gg p^{1/3}$  and  $|B| \gg \delta^{-1}$ , then we find a set  $\tilde{B} \subseteq B$  such that  $\mathbf{E}^\times(\tilde{B}) \lesssim |\tilde{B}|^{5/2}$  and  $|\tilde{B}| \geq |B|/2$ .

## 6 The proof of the main result

Using the results of the previous two parts of our paper, we are ready to formulate the main technical proposition of this section.

**Proposition 27** *Let  $A \subset \mathbb{F}_p$  be a set,  $\delta = |A|/p$ , and  $\varepsilon \in (0, 1]$  be a real number. Then for an arbitrary  $B \subseteq \text{Spec}_\varepsilon(A)$ ,  $|B| < \delta^{-1/6}\varepsilon^{-2/3}\sqrt{p}$  and any sets  $C, D \subseteq \mathbb{F}_p$ , one has*

$$\sum_{x \in D} (B \otimes C)(x) \lesssim \delta^{-1/6}\varepsilon^{-2/3}|B|^{1/2} \min\{|D|^{1/2}(\mathbf{E}^\times(C))^{1/4}, |C|^{1/2}(\mathbf{E}^\times(D))^{1/4}\}. \quad (29)$$

Further suppose that  $|C| \leq |D|$  and

$$|B| \leq \delta\varepsilon^4|C|^3|D|^2 \leq |B|^9 \quad (30)$$

as well as

$$\delta^{-1/4}\varepsilon^{-1}|B|^{9/4}|D|^{-1/2}|C|^{5/4} < p^2. \quad (31)$$

Then

$$\sum_{x \in D} (B \otimes C)(x) \lesssim \delta^{-3/16}\varepsilon^{-3/4}|B|^{11/16}|D|^{1/8}|C|^{15/16}. \quad (32)$$

Finally, assuming  $|C| \leq |D| < p^{2/3}$ ,  $|B| < \min\{\delta^{-1/6}\varepsilon^{-2/3}\sqrt{p}, p^{2/3}\}$  and

$$|C|^{20} \leq |D|^{45}\delta^{15}\varepsilon^{60}|B|^{24}, \quad (33)$$

we get

$$\sum_{x \in C} (B \otimes D)^2(x) \lesssim \min\{\delta^{-1/3}\varepsilon^{-4/3}|B|(\mathbf{E}^\times(D))^{1/2}, \delta^{-4/11}\varepsilon^{-16/11}|C|^{9/11}|B|^{29/22}|D|^{9/22}\}. \quad (34)$$

**Proof.** Let  $\sigma = \sum_{x \in D} (B \otimes C)(x)$ . Using the Cauchy–Schwarz inequality twice, combining with Theorem 20, we get

$$\sigma^4 \leq |D|^2 \mathbf{E}^\times(B) \mathbf{E}^\times(C) \lesssim |D|^2 \mathbf{E}^\times(C) |B|^2 \cdot \delta^{-2/3}\varepsilon^{-8/3}$$

and bound (29) has proved.

Now let us prove (32). First of all, notice that, trivially,  $\sigma \leq |B||C|$  and hence we can suppose

$$|B||C| > \delta^{-3/16}\varepsilon^{-3/4}|B|^{11/16}|D|^{1/8}|C|^{15/16}$$

or, in other words,

$$|B|^5|C|\varepsilon^{12}\delta^3 > |D|^2. \quad (35)$$

Finally, because of  $B \subseteq \text{Spec}_\varepsilon(A)$  we have in view of (22)

$$|B| \leq \delta^{-1}\varepsilon^{-2}. \quad (36)$$

Now let  $M \geq 1$  be a parameter which we will choose later. Our arguments is a sort of an algorithm. We construct a decreasing sequence of sets  $U_1 = B \supseteq U_2 \supseteq \dots \supseteq U_k$  and an increasing sequence of sets  $V_0 = \emptyset \subseteq V_1 \subseteq \dots \subseteq V_{k-1} \subseteq B$  such that for any  $j = 1, 2, \dots, k$  the sets  $U_j$  and  $V_{j-1}$  are disjoint and moreover  $B = U_j \sqcup V_{j-1}$ . If at some step  $j$  we have  $E^+(U_j) \leq |B|^3/M$ , then we stop our algorithm putting  $U = U_j$ ,  $V = V_{j-1}$ , and  $k = j - 1$ . In the opposite situation we have  $E^+(U_j) > |B|^3/M$ . Using the pigeonhole principle we find a set  $P_j \subseteq U_j - U_j$  such that  $E_{P_j}^+(U_j) \sim E^+(U_j)$  and a number  $t = t_j$  with  $t < (U_j \circ U_j)(x) \leq 2t$  for all  $x \in P_j$ . Applying Lemma 18 to the sets  $U_j$ ,  $P_j$ , we get the subset  $Y_j$  of  $U_j$  such that  $|Y_j| \gtrsim |B|/M$  and a number  $q_j \lesssim |Y_j|$  such that for any  $x \in Y_j$  one has  $(U_j * P_j)(x) \geq q_j$  and  $\sigma_{P_j}(U_j) \sim |Y_j|q_j$ . After that we put  $U_{j+1} = U_j \setminus Y_j$ ,  $V_j = V_{j-1} \sqcup Y_j$  and repeat the procedure. Clearly,  $V_k = \bigsqcup_{j=1}^k Y_j$  and because of  $|Y_j| \gtrsim |B|/M$ , we have  $k \lesssim M$ , so the number of steps is finite.

Consider  $\sigma_j = \sum_{x \in D} (Y_j \otimes C)(x)$ . By the Cauchy-Schwarz inequality and the fact that  $(U_j * P_j)(x) \geq q_j$  on  $Y_j$ , we have

$$\sigma_j^2 \leq |D| E^\times(Y_j, C) \leq q_j^{-2} |D| |\{(u + p)c = (u' + p')c' : u, u' \in U_j, p, p' \in P_j, c, c' \in C\}|.$$

Applying Theorem 14, we get

$$\sigma_j^2 \ll q_j^{-2} |D| ((|U_j||P_j||C|)^{3/2} + |U_j||P_j||C| \cdot \max\{|U_j|, |P_j|, |C|\}), \quad (37)$$

provided

$$|U_j||P_j||C| \ll p^2. \quad (38)$$

We will check condition (38) later. Moreover suppose that the first term in (37) dominates. Then using the fact  $q_j|Y_j| \sim t|P_j|$ , we obtain

$$\sigma_j^2 \ll q_j^{-2} |D| ((|U_j||P_j||C|)^{3/2} \lesssim |D||C|^{3/2}|B|^{3/2}|Y_j|^2 t^{-2} |P_j|^{-1/2}.$$

Now recalling that  $q_j \lesssim |Y_j|$  and observing

$$t|Y_j|^2 \gtrsim tq_j|Y_j| \sim t\sigma_{P_j}(U_j) \sim t^2|P_j| \sim E^+(U_j),$$

we get from  $t|Y_j|^2 \gtrsim t^2|P_j|$  and  $t|Y_j|^2 \gtrsim E^+(U_j)$  that  $E^+(U_j)/|Y_j|^2 \lesssim t \lesssim |Y_j|^2/|P_j|$  and hence in view of  $t^2|P_j| \sim E^+(U_j)$ , we derive

$$|P_j| \lesssim |Y_j|^4 / E^+(U_j) \quad (39)$$

and

$$\begin{aligned} \sigma_j^2 &\lesssim |D||C|^{3/2}|B|^{3/2}|Y_j|^2 E^+(U_j)^{-1} |P_j|^{1/2} \lesssim |D||C|^{3/2}|B|^{3/2}|Y_j|^4 E^+(U_j)^{-3/2} \leq \\ &\leq M^{3/2} |D||C|^{3/2}|Y_j|^4 |B|^{-3}. \end{aligned}$$

Thus

$$\sigma = \sum_{x \in D} (U \otimes C)(x) + \sum_{x \in D} (V \otimes C)(x) \leq \sum_{x \in D} (U \otimes C)(x) + \sum_{j=1}^k \sigma_j \lesssim \quad (40)$$

$$\lesssim \sum_{x \in D} (U \otimes C)(x) + M^{3/4} |D|^{1/2} |C|^{3/4} |B|^{-3/2} \sum_{j=1}^l |Y_j|^2 \leq \quad (41)$$

$$\leq \sum_{x \in D} (U \otimes C)(x) + M^{3/4} |D|^{1/2} |C|^{3/4} |B|^{1/2}. \quad (42)$$

To estimate the first term in the last formula, we remind that  $\mathbf{E}^+(U) \leq |B|^3/M$  and  $U \subseteq B \subseteq \text{Spec}_\varepsilon(A)$ . Using Lemma 17, we see that

$$\delta \varepsilon^4 |U|^4 \leq \mathbf{E}^+(U) \leq |B|^3/M.$$

Whence

$$|U| \leq \delta^{-1/4} \varepsilon^{-1} M^{-1/4} |B|^{3/4} \quad (43)$$

and thus

$$\sigma \lesssim \delta^{-1/4} \varepsilon^{-1} M^{-1/4} |B|^{3/4} \cdot \min\{|C|, |D|\} + M^{3/4} |D|^{1/2} |C|^{3/4} |B|^{1/2}.$$

Recall that  $m := \min\{|C|, |D|\} = |C|$ . The optimal choice of  $M$  is

$$M = \delta^{-1/4} \varepsilon^{-1} |B|^{1/4} |D|^{-1/2} |C|^{-3/4} m = \delta^{-1/4} \varepsilon^{-1} |B|^{1/4} |D|^{-1/2} |C|^{1/4}$$

and hence

$$\sigma \lesssim \delta^{-3/16} \varepsilon^{-3/4} m^{3/4} |B|^{11/16} |D|^{1/8} |C|^{3/16} = \delta^{-3/16} \varepsilon^{-3/4} |B|^{11/16} |D|^{1/8} |C|^{15/16}.$$

It is easy to see that the inequality  $M \geq 1$  is equivalent to

$$|B||C| \geq \delta \varepsilon^4 |D|^2$$

but in view of (35) it would follow from

$$|B|^4 \leq \delta^{-4} \varepsilon^{-16}.$$

The last inequality is a simple consequence of (36). Now let us check condition (38). In view of estimate (39) it is sufficient to have

$$|U_j||P_j||C| \lesssim |Y_j|^4 |U_j||C|/\mathbf{E}^+(U_j) \leq M|C||B|^2 = \delta^{-1/4} \varepsilon^{-1} |B|^{9/4} |D|^{-1/2} |C|^{5/4} \ll p^2.$$

The last bound is our condition (31) (again we ignore signs  $\ll, \gg$  increasing the constants in the final inequalities as in the proof of Lemma 15).

It remains to consider the case when the second term in (37) dominates. We will show that in this situation one has even better upper bound for  $\sigma$ . Put  $\nu_j = \max\{|U_j|, |P_j|, |C|\}$ . In view of formulas (37), (40)—(42) and our choice of  $q_j$ , it is sufficient to check

$$|D|^{1/2} \sum_j q_j^{-1} (|U_j||P_j||C|\nu_j)^{1/2} \lesssim |D|^{1/2} \sum_j t_j^{-1} |Y_j| (|U_j||P_j|^{-1} |C|\nu_j)^{1/2} \lesssim M^{3/4} |D|^{1/2} |C|^{3/4} |B|^{1/2}.$$

If  $\nu_j = |P_j|$ , then we obtain the inequality to insure

$$\sum_j |Y_j| t_j^{-1} |U_j|^{1/2} \lesssim M^{3/4} |C|^{1/4} |B|^{1/2}.$$



Clearly,  $|U_j| \leq |B|$ ,  $t_j \gtrsim |B|/M$  and  $\sum_j |Y_j| \leq |B|$ . Thus we need to check

$$M \leq |C|$$

or, in other words,

$$|B| \leq |C|^3 |D|^2 \delta \varepsilon^4$$

and this is the first part of condition (30). If  $\nu_j = U_j$ , then we have the bound

$$\sum_j |Y_j| |U_j| (t_j^2 |P_j|)^{-1/2} \lesssim \sum_j (\mathbf{E}^+(B))^{-1/2} |Y_j| |U_j| \leq (|B|^3/M)^{-1/2} \sum_j |Y_j| |U_j| \leq M^{1/2} |B|^{1/2}.$$

Clearly, the last quantity is less than  $M^{3/4} |C|^{1/4} |B|^{1/2}$ . Finally, if  $\nu_j = |C|$ , then similarly, we get

$$\sum_j |Y_j| |U_j|^{1/2} |C|^{1/2} (t_j^2 |P_j|)^{-1/2} \lesssim M^{1/2} |C|^{1/2}.$$

To make this less than  $M^{3/4} |C|^{1/4} |B|^{1/2}$  it is sufficient to have

$$|C| \leq M |B|^2$$

or

$$|C|^3 |D|^2 \delta \varepsilon^4 \leq |B|^9.$$

The last inequality coincides with the second part of conditions (30). Thus, we have proved the second part of our proposition.

It remains to obtain (34). The first bound is a trivial consequence of the Cauchy–Schwarz inequality, combining with Theorem 20. Here we simply ignore that the summation is taken over the set  $C$ . Notice that we do not use condition (33) as well as  $|C| \leq |D| < p^{2/3}$ ,  $|B| < p^{2/3}$  to obtain this bound but the assumption  $|B| < \delta^{-1/6} \varepsilon^{-2/3} \sqrt{p}$  only. Let us prove the second estimate, where we need all mentioned assumptions. In our arguments we apply the algorithm above and construct the sets  $U, V$ ,  $U \sqcup V = B$ , in particular. Using the Cauchy–Schwarz inequality, Lemma 12 and bound (43), we get

$$\begin{aligned} \sum_{x \in C} (B \otimes D)^2(x) &= \sum_{x \in C} ((U \otimes D)(x) + (V \otimes C)(x))^2 \ll \sum_{x \in C} (U \otimes D)^2(x) + \sum_{x \in C} (V \otimes D)^2(x) \leq \\ &\leq |U|^2 |C| + \left( \sum_{j=1}^k (\mathbf{E}_C^\times(Y_j, D))^{1/2} \right)^2 \lesssim \delta^{-1/2} \varepsilon^{-2} M^{-1/2} |B|^{3/2} |C| + \sigma_*. \end{aligned}$$

Here  $M$  is a parameter which we will choose later. Our task is to find a good upper bound for  $\sigma_*$ . To estimate the sum  $\sigma_*$  we need to bound  $\mathbf{E}_C^\times(Y_j, D)$  via Corollary 16 with  $A = Y_j$ ,  $X = C$ ,  $Y = D$ ,  $P = P_j$ ,  $Q = U_j$  and  $t = q_j$ . To apply this corollary we have to find the condition on the parameter  $M$  when

$$|P_j|^2 |U_j|^{5/4} \geq q_j^2 |Y_j|^{3/4}. \quad (44)$$

Suppose not. Then by formula  $q_j |Y_j| \sim t_j |P_j|$ , we obtain

$$|P_j|^2 |U_j|^{5/4} \lesssim t_j^2 |Y_j|^{-5/4} |P_j|^2$$

and because of  $q_j \lesssim |Y_j|$ , we have  $|Y_j|^2 \gtrsim |Y_j|q_j \sim |P_j|t_j$  and hence

$$|U_j|^{5/4}(|P_j|t_j)^{5/8} \lesssim |U_j|^{5/4}|Y_j|^{5/4} \lesssim t_j^2.$$

This implies

$$|U_j|^{10}|B|^{15}M^{-5} \lesssim |U_j|^{10}\mathbf{E}^+(U_j)^5 \lesssim t_j^{21} \leq |U_j|^{21}.$$

Thus (44) takes place if

$$M \lesssim |B|^{4/5}. \quad (45)$$

In this case the conditions of Corollary 16 take place because  $|D| < p^{2/3}$  and  $|U_j| \leq |B| < p^{2/3}$ . Applying this corollary, formulas  $q_j|Y_j| \sim t|P_j|$ ,  $t_j^2|P_j| \sim \mathbf{E}^+(U_j)$  and inequality (39), we obtain

$$\begin{aligned} \sigma_* &\leq |D|^{3/2}|C|^{1/3} \left( \sum_{j=1}^k q_j^{-1}|P_j|^{2/3}|U_j|^{3/4} \right)^2 \lesssim |D|^{3/2}|C|^{1/3}|B|^{3/2} \left( \sum_{j=1}^k |Y_j|t_j^{-1}|P_j|^{-1/3} \right)^2 \lesssim \\ &\lesssim |D|^{3/2}|C|^{1/3}|B|^{3/2} \left( \sum_{j=1}^k |Y_j||P_j|^{1/6}(\mathbf{E}^+(U_j))^{-1/2} \right)^2 \lesssim \\ &\lesssim |D|^{3/2}|C|^{1/3}|B|^{3/2} \left( \sum_{j=1}^k |Y_j|^{5/3}(\mathbf{E}^+(U_j))^{-2/3} \right)^2 \leq M^{4/3}|D|^{3/2}|C|^{1/3}|B|^{5/6}. \end{aligned}$$

Thus

$$\sum_{x \in C} (B \otimes D)^2(x) \lesssim \delta^{-1/2}\varepsilon^{-2}M^{-1/2}|B|^{3/2}|C| + M^{4/3}|D|^{3/2}|C|^{1/3}|B|^{5/6}.$$

The optimal choice of  $M$  is

$$M = |B|^{4/11}|C|^{4/11}\delta^{-3/11}\varepsilon^{-12/11}|D|^{-9/11}$$

and hence

$$\sum_{x \in C} (B \otimes D)^2(x) \lesssim \delta^{-4/11}\varepsilon^{-16/11}|C|^{9/11}|B|^{29/22}|D|^{9/22}$$

as required. It remains to notice that the condition  $M \lesssim |B|^{4/5}$  is equivalent to (33). This completes the proof.  $\square$

Bound (32) works better than (29) or (34) in the case when the size of  $D$  is large comparable to  $B$  and  $C$ . For very small  $D$  estimate (29) is the best one.

Let us prove our main result.

**Theorem 28** *Let  $\mathcal{E}$  be a finite family of equations of form (12). Also, let  $A \subseteq \mathbb{F}_p$  be a set avoiding the family  $\mathcal{E}$  and  $|A| \gg p^{\frac{39}{47}}$ . Then for any  $\kappa_1 < \frac{10}{31}$ , one has*

$$|A| \ll \frac{p}{\mathcal{T}(\mathcal{E})^{\kappa_1}}. \quad (46)$$

and for an arbitrary  $\kappa_2 < \frac{3}{10}$  the following holds

$$|A| \ll \frac{p}{\mathcal{T}_*(\mathcal{E})^{\kappa_2}}. \quad (47)$$

Finally, let  $|A| \gg p^{7/9}$ ,  $\mathcal{T}_*(\mathcal{E}) < p^{2/3}$ . Then for an arbitrary  $\kappa_3 < \frac{35}{159}$  one has

$$|A| \ll \max \left\{ \frac{p}{\mathcal{T}_*(\mathcal{E})^{\kappa_3}} \cdot \left( \frac{\mathbf{E}_*^+(A)}{|A|^3} \right)^{\frac{22}{159}}, \frac{p}{\mathcal{T}_*(\mathcal{E})^{69/183}} \right\}. \quad (48)$$

**Proof.** Let  $|A| = \delta p$  and  $t = \mathcal{T}(\mathcal{E})$ . By assumption the set  $A$  avoids all equations from the family  $\mathcal{E}$ . Using the Fourier transform, we see that it is equivalent to

$$0 = \sum_r \widehat{A}(a_j r) \widehat{A}(b_j r) \widehat{A}(c_j r) e(-d_j r) = |A|^3 + \sum_{r \neq 0} \widehat{A}(a_j r) \widehat{A}(b_j r) \widehat{A}(c_j r) e(-d_j r) \quad (49)$$

for all  $j \in [t]$ . Applying Parseval identity (6) three times, we have

$$2^{-1}|A|^3 \leq \sum_{r \in (a_j^{-1}B) \cap (b_j^{-1}B) \cap (c_j^{-1}B)} |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| |\widehat{A}(c_j r)|, \quad (50)$$

where  $B = \text{Spec}_\varepsilon(A) \setminus \{0\}$ ,  $\varepsilon = \delta/6$ . Indeed,

$$\begin{aligned} \sum_{r \notin c_j^{-1}B} |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| |\widehat{A}(c_j r)| &\leq \varepsilon |A| \sum_r |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| \\ &\leq \varepsilon |A| \left( \sum_r |\widehat{A}(a_j r)|^2 \right)^{1/2} \left( \sum_r |\widehat{A}(b_j r)|^2 \right)^{1/2} \leq \varepsilon |A| |A| p = |A|^3/6 \end{aligned}$$

and similar for another two terms. Here we have used that  $a_j, b_j, c_j$  are nonzero numbers.

Let  $S := \{s_1, \dots, s_t\} \subseteq S(\mathcal{E})$  such that, say,  $s_j = (a_j, b_j, 1)$ , where  $a_j, b_j$  are different. In particular,  $a_j, b_j$  belong to two sets  $S_A, S_B$ , correspondingly, and  $|S_A| = |S_B| = t$ . Now let us return to (50). Summing the last estimate over  $S$ , and using the Cauchy–Schwartz inequality and Parseval identity (6), we have

$$\begin{aligned} t^2 |A|^6 &\ll \left( \sum_{r \in B} |\widehat{A}(r)| \sum_{j=1}^t |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| B(a_j r) B(b_j r) \right)^2 \ll \\ &\ll \sum_r |\widehat{A}(r)|^2 \cdot \sum_{r \in B} \left( \sum_{j=1}^t |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| B(a_j r) B(b_j r) \right)^2 \leq \\ &\leq |A| p \cdot \sum_{r \in B} \left( \sum_{j=1}^t |\widehat{A}(a_j r)| |\widehat{A}(b_j r)| B(a_j r) B(b_j r) \right)^2. \end{aligned}$$

Using the pigeonholing principle twice, we find two numbers  $\Delta_1, \Delta_2 \leq |A|$  and two sets  $W_1, W_2 \subseteq B$  such that  $\Delta_1 < |\hat{A}(r)| \leq 2\Delta_1$  for  $r \in W_1$ ,  $\Delta_2 < |\hat{A}(r)| \leq 2\Delta_2$  for  $r \in W_2$  and

$$\begin{aligned} t^2|A|^6 &\lesssim |A|p\Delta_1^2\Delta_2^2 \cdot \sum_{r \in B} \left( \sum_{j=1}^t W_1(a_j r) W_2(b_j r) \right)^2 \leq \\ &\leq |A|p\Delta_1^2\Delta_2^2 \cdot \sum_{r \in B} (S_A^{-1} \otimes W_1)(r) (S_B^{-1} \otimes W_2)(r). \end{aligned} \quad (51)$$

Put  $\varepsilon_1 = \Delta_1/|A|$ ,  $\varepsilon_2 = \Delta_2/|A|$ . In particular, from formula (51), combining with (6), we get

$$t^2|A|^6 \lesssim |A|p\Delta_1^2\Delta_2^2|W_1||W_2||B| \leq (|A|p)^3|B| \quad (52)$$

and hence

$$|B| \gtrsim \delta^3 t^2. \quad (53)$$

as well as

$$|B||W_1| \gtrsim \delta^2 t^2 \quad \text{and} \quad |B||W_2| \gtrsim \delta^2 t^2. \quad (54)$$

In particular, in view of  $|B| \ll \delta^{-3}$ , we obtain

$$|W_1|, |W_2| \gtrsim \delta^5 t^2. \quad (55)$$

The singes  $\lesssim, \gtrsim$  in formulas (52)–(55) as well as in all formulas below depend on the size of the set  $|B|$ . The last quantity is less than  $O(\delta^{-3})$  and so it depends on the density of the set  $A$  but on the size of  $A$ . Thus we can remove these logarithms requiring strictly smaller power of  $\delta$  in the formulation of the theorem.

Now put  $m = \min\{|W_1|, |W_2|\}$  and let  $m = |W_2|$  for certainty. Further we have

$$t^2|A|^6 \lesssim |A|p\Delta_1^2\Delta_2^2 m \cdot \sum_{r \in B} (S_A^{-1} \otimes W_1)(r).$$

As above by (22) we see that  $|B| \ll \delta^{-3}$ , so if  $\delta^{-3} \gg t$ , then  $\delta \ll t^{-1/3}$  and it is nothing to prove. Hence one can assume that  $|W_1|, |W_2|, |B| \leq t$  (again we ignore signs  $\ll, \gg$  increasing the constants in the final inequalities as in Lemma 15). Split the set  $B$  into some  $s$  sets  $B_j$  of approximately equal sizes, where  $s$  is a parameter which we will choose later. Using the bound  $|B| \ll \delta^{-3}$  and applying the Parseval identity one more time as well as the second part of Proposition 27 with  $\varepsilon = \varepsilon_1 = \Delta_1/|A|$ ,  $B = W_1$ ,  $C = B_j^{-1}$ ,  $D = S_A$ , we obtain

$$t^2 \delta^6 p^6 = t^2|A|^6 \lesssim |A|p\Delta_1^2\Delta_2^2 m \sum_{j=1}^s \sum_{r \in B_j} (S_A^{-1} \otimes W_1)(r) \lesssim \quad (56)$$

$$\lesssim |A|p\Delta_1^2\Delta_2^2 m \delta^{-3/16} (\Delta_1/|A|)^{-3/4} t^{1/8} |W_1|^{11/16} \sum_{j=1}^s |B_j|^{15/16} \ll \quad (57)$$

$$\ll |A|^2 p^2 \delta^{-3} |A|^{3/4} \Delta_1^{5/4} |W_1|^{11/16} t^{1/8} s^{1/16} \leq \delta^{-1/4} p^{19/4} (\Delta_1^2 |W_1|)^{5/8} |W_1|^{1/16} t^{1/8} s^{1/16} \leq \quad (58)$$

$$\leq p^6 \delta^{3/8} |W_1|^{1/16} t^{1/8} s^{1/16}. \quad (59)$$

It remains to check that all conditions of Proposition 27 satisfy and choose the parameter  $s$ . We have already insured that  $|C| \leq |D|$  since  $|W_1|, |W_2|, |B| \leq t$ . If

$$\begin{aligned} p^2 &< \delta^{-1/4} \varepsilon_1^{-1} |W_1|^{9/4} t^{-1/2} |B_j|^{5/4} \leq \delta^{-1/4} \varepsilon_1^{-1} |W_1|^{9/4} t^{-1/2} |B|^{5/4} \ll \\ &\ll \delta^{-1/4-1-3.7/2} = \delta^{-47/4}, \end{aligned} \quad (60)$$

then one can easily arrives to a contradiction with the assumption  $|A| \gg p^{39/47}$ . Thus condition (31) takes place. Further from (53) for any  $j$ , it follows that

$$\delta \varepsilon_1^4 |B_j|^3 t^2 \gg \delta^5 |B|^3 t^2 s^{-3} \geq \delta^5 |W_1| |B|^2 t^2 s^{-3} \gtrsim \delta^{11} |W_1| t^6 s^{-3} \geq |W_1|$$

provided  $s \leq \delta^{11/3} t^2$ . Further

$$\delta \varepsilon_1^4 |B_j|^3 t^2 \ll \delta |B|^3 t^2 s^{-3} \ll \delta^{-8} t^2 s^{-3} \leq |W_1|^9$$

provided  $s \gg \delta^{-8/3} t^{2/3} |W_1|^{-3}$ . Putting  $s = \delta^{-8/3} t^{2/3} |W_1|^{-3}$  one can insure that  $s \ll \delta^{11/3} t^2$  because otherwise in view of (55), we have

$$\delta^{-8} t^2 \gtrsim \delta^{11} t^6 |W_1|^9 \gtrsim \delta^{56} t^{24}$$

or, in other words,  $\delta \lesssim t^{-11/32}$  which is better than (46). If  $s \ll 1$ , then from (59), we obtain

$$t^{15/8} \delta^{45/8} \lesssim |W_1|^{1/16} \ll \delta^{-3/16}$$

or

$$\delta \lesssim t^{-10/31}.$$

The last bound coincides with (46). Finally, we should note that in the case  $s \ll 1$  one quickly insure that the condition  $\delta \varepsilon_1^4 |B|^3 t^2 \gtrsim |W_1|$  takes place. Thus from (59), (55) and our choice of the parameter  $s$ , it follows that

$$t^2 \delta^6 \lesssim \delta^{3/8} t^{1/8} \delta^{-1/6} t^{1/24} |W_1|^{-1/8} \lesssim \delta^{3/8} t^{1/8} \delta^{-1/6} t^{1/24} \cdot \delta^{-5/8} t^{-1/4}$$

or

$$\delta \lesssim t^{-25/77}$$

which is better than (46) again.

Now let us prove the second part of the theorem. Put  $t_* = \mathcal{T}_*(\mathcal{E})$  and let  $S_* = \{s_1, \dots, s_{t_*}\}$  be the set from the Definition 5 (without losing of the generality we consider the intersection of  $S(\mathcal{E})$  with the plane  $\{z = 1\}$ ). Returning to (50) and then after changes of variables, we have for any  $j \in [t_*]$  that

$$2^{-1} |A|^3 \leq \sum_{r \in B \cap (B/s_j) \cap (B/s'_j)} |\hat{A}(s_j r)| |\hat{A}(s'_j r)| |\hat{A}(r)|.$$

Here  $s'_j = a_j$  if  $s_j = b_j$ , further  $s'_j = b_j$  if  $s_j = a_j$  and, finally,  $s'_j = b_j^{-1}$  if  $s_j = a_j/b_j$ . Since  $s'_j \neq 0$ , we obtain by the Cauchy–Schwarz inequality and formula (6)

$$|A|^6 \ll |A|p \sum_{r \in B \cap (B/s_j)} |\hat{A}(s_j r)|^2 |\hat{A}(r)|^2. \quad (61)$$

Thus, summing over  $j \in [t_*]$ , we get

$$|A|^5 t_* \ll p \sum_{j=1}^{t_*} \sum_{r \in B \cap (B/s_j)} |\hat{A}(s_j r)|^2 |\hat{A}(r)|^2. \quad (62)$$

Using the pigeonholing principle twice, we find two numbers  $\Delta_1, \Delta_2 \leq |A|$  and two sets  $W_1, W_2 \subseteq B$  such that  $\Delta_1 < |\hat{A}(r)| \leq 2\Delta_1$  for  $r \in W_1$ ,  $\Delta_2 < |\hat{A}(r)| \leq 2\Delta_2$  for  $r \in W_2$  and

$$|A|^5 t_* \lesssim p \Delta_1^2 \Delta_2^2 \sum_r W_2(r) (W_1 \otimes S_*^{-1})(r). \quad (63)$$

As above, we have  $|W_1|, |W_2| \leq t_*$  because otherwise it is nothing to prove. Similarly, one can check that the conditions

$$\delta^{-1/4} \varepsilon_1^{-1} |W_1|^{9/4} t_*^{-1/2} |W_2|^{5/4} < p^2, \quad \delta^{-1/4} \varepsilon_2^{-1} |W_2|^{9/4} t_*^{-1/2} |W_1|^{5/4} < p^2.$$

follows from the assumption  $|A| \gg p^{39/47}$ . Here  $\varepsilon_1 = \Delta_1/|A|$  and  $\varepsilon_2 = \Delta_2/|A|$ . Further from (63), we obtain

$$t_* \delta |A|^4 \ll \Delta_1^2 \Delta_2^2 |W_1| |W_2|.$$

Whence in view of the Parseval identity, we have

$$|W_1|, |W_2| \gg \delta^2 t_* \quad \text{and} \quad |W_1| |W_2| \gg \delta t_*. \quad (64)$$

Suppose that  $|W_2| \geq |W_1|$  for certainty. Split the set  $W_2$  into some  $s$  sets  $W_2^{(j)}$  of approximately equal sizes, where  $s$  is a parameter which we will choose later. Bounds (64) imply for any  $j$

$$\delta \varepsilon_1^4 |W_2^{(j)}|^3 t_*^2 \gg \delta^5 |W_2|^3 t_*^2 s^{-3} \gg |W_1|$$

provided  $s \ll |W_2| t_*^{2/3} \delta^{5/3} |W_1|^{-1/3}$ . Further

$$\delta \varepsilon_1^4 |W_2^{(j)}|^3 t_*^2 \ll \delta |W_2|^3 t_*^2 s^{-3} \ll |W_1|^9$$

provided  $s \gg \delta^{1/3} |W_2| t_*^{2/3} |W_1|^{-3}$ . Putting  $s = \delta^{1/3} |W_2| t_*^{2/3} |W_1|^{-3}$  one can insure that  $s \ll |W_2| t_*^{2/3} \delta^{5/3} |W_1|^{-1/3}$  because otherwise in view of (64), we have

$$1 \gg |W_1|^{8/3} \delta^{4/3} \gg t_*^{8/3} \delta^{20/3}$$

or, in other words,  $\delta \lesssim t_*^{-2/5}$  which is better than (47). Suppose, in addition, that  $s \gg 1$ . Thus all conditions of the second part of Proposition 27 takes place. Applying arguments as in (56)–(59),

bounds (64) and using Proposition 27 with  $\varepsilon = \varepsilon_1$ ,  $B = W_1$ ,  $C = (W_2^{(j)})^{-1}$ ,  $D = S_*$  and the Parseval identity, we have

$$\begin{aligned}
|A|^5 t_*^{7/8} &\lesssim p \Delta_1^2 \Delta_2^2 \delta^{-3/16} (|A|/\Delta_1)^{3/4} |W_1|^{11/16} (|W_2|/s)^{15/16} s = \\
&= p |A|^{3/4} \Delta_1^{5/4} \Delta_2^2 \delta^{-3/16} |W_1|^{11/16} |W_2|^{15/16} s^{1/16} \leq \\
&\leq p |A|^{3/4} (p|A|)^{13/8} \delta^{-3/16} |W_1|^{1/16} |W_2|^{-1/16} s^{1/16} = p^{21/8} |A|^{19/8} \delta^{-3/16} |W_1|^{1/16} |W_2|^{-1/16} s^{1/16} \\
&\ll p^5 \delta^{35/16} (\delta^{1/3} t_*^{2/3} |W_1|^{-2})^{1/16} \ll p^5 \delta^{35/16} (\delta^{-11/3} t_*^{-4/3})^{1/16}.
\end{aligned} \tag{65}$$

It gives us

$$\delta \lesssim t_*^{-23/73} \tag{67}$$

which is better than (47). If  $s \ll 1$  then from (66) and (64), we see that

$$\delta^5 t_*^{7/8} \lesssim \delta^{35/16} |W_1|^{1/16} |W_2|^{-1/16} \lesssim \delta^2 (\delta^2 t_*)^{-1/16} = \delta^{15/8} t_*^{-1/16}$$

or, in other words,

$$\delta \lesssim t_*^{3/10}$$

which coincides with (47). Finally, we should note that in the case  $s \ll 1$  in view of the inequality  $|W_2| \geq |W_1|$  and bound (64), we easily have

$$\delta \varepsilon_1^4 |W_2|^3 t_*^2 \gg \delta^5 |W_1| |W_2|^2 t_*^2 \gg \delta^9 |W_1| t_*^4 \gg |W_1|$$

because otherwise we obtain  $\delta \lesssim t_*^{-4/9}$  which is much better than (47).

It remains to prove the last part of the theorem. Returning to (62) and squaring, we obtain

$$|A|^{10} t_*^2 \lesssim p^3 \mathbf{E}_*^+(A) \Delta^4 \sum_{r \in B} (W \otimes S_*^{-1})^2(r).$$

Here  $\Delta := \varepsilon |A| \leq |A|$  and  $W \subseteq B$  comes from the pigeonhole principle as above. Notice that the condition  $|A| \gg p^{7/9}$  implies

$$|B| \leq (\delta \varepsilon^2)^{-1} \ll \delta^{-3} < p^{2/3}.$$

Further the condition (recall the inequality  $|W| \gtrsim \delta^2 t_*$ )

$$t_*^{45} \delta^{15} \varepsilon^{60} |W|^{24} \gg t_*^{45} \delta^{75} |W|^{24} \gtrsim t_*^{69} \delta^{123} \gg |B|^{20}$$

trivially holds because otherwise

$$\delta \ll t_*^{-69/183}.$$

Hence, applying the third part of Proposition 27 with  $C = B$ ,  $B = W$ ,  $D = S_*^{-1}$ ,  $\varepsilon = \Delta/|A|$  and the Parseval identity, we get

$$|A|^{10} t_*^{35/22} \lesssim p^3 \mathbf{E}_*^+(A) \Delta^4 \delta^{-4/11} (|A|/\Delta)^{16/11} |B|^{9/11} |W|^{29/22} =$$

$$= p^3 \mathbf{E}_*^+(A) |A|^{16/11} \Delta^{28/11} \delta^{-4/11} |B|^{9/11} |W|^{29/22} \leq p^3 \mathbf{E}_*^+(A) |A|^{16/11} (|A|p)^{28/22} \delta^{-4/11} |B|^{9/11} |W|^{1/22}.$$

Using  $|B|, |W| \ll \delta^{-3}$ , we have

$$\delta^{159/22} \lesssim \mathbf{E}_*^+(A) / |A|^3 \cdot t_*^{-35/22}$$

or

$$\delta \lesssim t_*^{-35/159} \cdot (\mathbf{E}_*^+(A) / |A|^3)^{22/159}.$$

This completes the proof.  $\square$

In view of Lemma 6, we obtain

**Corollary 29** *Let  $\mathcal{E}$  be a finite family of equations of form (12). Also, let  $A \subseteq \mathbb{F}_p$  be a set avoiding the family  $\mathcal{E}$ ,  $|A| \gg p^{\frac{39}{47}}$ . Then for any  $\kappa < \frac{5}{31}$  one has*

$$|A| \ll \frac{p}{|\mathcal{E}|_\kappa}.$$

If one use the parameter  $s$  in estimate (60), then the restriction  $|A| \gg p^{\frac{39}{47}}$  in the first two parts of Theorem 28 as well as in Corollary 29 can be refined. We do not make such calculations.

Clearly, Proposition 9, combining with Theorem 28 give Theorem 1 from the introduction. Further it is easy to see that  $\mathbf{E}^+(A) = o(|A|^3)$  implies that any set avoiding just *one* equation has size  $o(p)$ . Inequality (48) can be considered as a generalization of this fact for several equations.

**Remark 30** *It is easy to see from the proof of Theorem 28 that the same arguments work for sets having, say, at most  $|A|^3/(4p)$  or at least  $2|A|^3/p$  solutions of equations (12). In other words, the number of solutions must differ from the expectation significantly.*

## 7 Further applications

This section contains three applications of the results above. Let us consider the first one.

In [13], [16] authors studied a family of subsets of  $\mathbb{Z}$  which generalize arithmetic progressions of length three. Let us recall the definition. Let  $t \geq 1$  be a fixed integer. A finite set  $A \subset \mathbb{Z}$  is called *non-averaging of order  $t$* , if for every  $1 \leq m, n \leq t$  the equation

$$mX_1 + nX_2 = (m+n)X_3 \tag{68}$$

have just trivial solutions:  $X_1 = X_2 = X_3$ . For example, if  $t = 1$ , then  $A$  is non-averaging of order 1 iff  $A$  has no arithmetic progressions of length three. The best upper bound for the size of a subset of  $[N]$  having no arithmetic progressions of length three as well the history of the question can be found in [3]. Namely, developing the method of Sanders [14], T.F. Bloom proved that

$$|A| \ll \frac{N(\log \log N)^4}{\log N}. \tag{69}$$



Here we obtain a new upper bound for the size of a non-averaging set of order  $t$  in  $\mathbb{F}_p$ , that is, a set having no non-trivial solutions of system (68) in  $\mathbb{F}_p$ . It is known that the modular version of the question about the density of arithmetic progressions is equivalent to the integer case. In particular, inequality (69) takes place with  $N = p$  for sets  $A \subseteq \mathbb{F}_p$  without solutions  $x + y \equiv 2z \pmod{p}$ .

**Theorem 31** *Let  $A \subseteq \mathbb{F}_p$  be a non-averaging set of order  $t$ ,  $t < \sqrt{p}$ . Then for any  $\kappa < 20/31$  one has*

$$|A| \ll \max \left\{ \frac{p}{t^\kappa}, p^{39/47} \right\}. \quad (70)$$

**Proof.** By our assumption the set  $A$  avoids all equations from (68). In other words,  $X_1 + n/m \cdot X_2 = (1 + n/m)X_3$ , where  $X_1, X_2, X_3 \in A$  implies  $X_1 = X_2 = X_3$ . Thus, we have the correspondent system  $\mathcal{E}$  with the set  $S(\mathcal{E})$  of cardinality  $||t|/[t]|$ . In a similar way

$$\mathcal{T}(\mathcal{E}) = |\{n/m : n, m \in [t]\}| = |[t]/[t]|.$$

Considering square-free numbers, it is easy to see in view of the assumption  $t < \sqrt{p}$  that  $||t|/[t]| \gg t^2$  both in  $\mathbb{Z}$  and in  $\mathbb{F}_p$  (consult the proof of Proposition 9). Although Theorem 28 was formulated just for sets having no solutions at all, it is easy to insure that the number of trivial solutions is  $|A|$ . Thus if  $|A|p \leq |A|^3/4$ , say, then the method of the proof works (see Remark 30). Of course, if  $|A|p > |A|^3/4$ , then  $|A| < 2\sqrt{p}$  and there is nothing to prove. Whence, applying the first part of Theorem 28, we obtain the required result.  $\square$

Thus, taking any  $C > 31/20$  and  $t$  such that  $t \geq (\log p)^C$ , we see that bound (70) is better than (69) in this case.

Now consider another application.

Let  $A \times A$  is the Cartesian product of a set  $A \subseteq \mathbb{F}_p$ . The number of *collinear triples*  $\mathsf{T}(A)$  in  $A \times A$  is an important characteristic of a set, see [1], [20], [21], say. Observe (or see [20], [21]) that

$$\mathsf{T}(A) = \left| \left\{ \frac{a_1 - a}{a'_1 - a'} = \frac{a_2 - a}{a'_2 - a'} : a_1, a_2, a, a'_1, a'_2, a' \in A \right\} \right|.$$

(we suppose in the formula above that  $a'_1 = a'$  implies  $a'_2 = a'$  and vice versa). Another formula for  $\mathsf{T}(A)$  is (see [20], [21] again)

$$\mathsf{T}(A) = \sum_{a, a' \in A} \mathsf{E}^\times(A - a, A - a') + O(|A|^4). \quad (71)$$

The quantity  $\mathsf{T}[A]$  is naturally connected with the set

$$R[A] := \left\{ \frac{a_1 - a}{a_2 - a} : a_1, a_2, a \in A, a_2 \neq a \right\}. \quad (72)$$

Namely,

$$\mathsf{T}(A) = \sum_{\lambda \in R[A]} q^2(\lambda) + O(|A|^4), \quad (73)$$

where

$$q(\lambda) = \left| \left\{ \frac{a_1 - a}{a_2 - a} = \lambda : a_1, a_2, a \in A, a_2 \neq a \right\} \right|.$$

In [1] authors obtained an upper bound for  $T(A)$  in the case of small sets  $A$ .

**Theorem 32** *Let  $A \subset \mathbb{F}_p$  be a set with  $|A| < p^{2/3}$ . Then*

$$T(A) \ll |A|^{9/2}.$$

Now we extend this result to larger sets, obtaining an asymptotic formula for the quantity  $T(A)$ .

**Theorem 33** *Let  $A \subseteq \mathbb{F}_p$  be a set,  $|A| > p^{2/5}$ . Then for some absolute constant  $C > 0$  the following holds*

$$\left| T(A) - \frac{|A|^6}{p} \right| \ll \log^C(p/|A|) \cdot |A|^{40/9} p^{2/9}. \quad (74)$$

**Proof.** Put  $|A| = a = \delta p$ ,  $q_*(\lambda) = q(\lambda) - a^3/p$ . Because of  $\sum_{\lambda} q(\lambda) = a^2(a-1)$ , we have

$$\sum_{\lambda} |q_*(\lambda)| \leq \sum_{\lambda} (q(\lambda) + a^3/p) \leq 2a^3, \quad (75)$$

$$\sum_{\lambda} q_*(\lambda) = -a^2,$$

and hence

$$\sum_{\lambda} q^2(\lambda) = \sum_{\lambda} \left( q_*(\lambda) + \frac{a^3}{p} \right)^2 = \sum_{\lambda} q_*^2(\lambda) + \frac{a^6}{p} + \frac{2a^3}{p} \sum_{\lambda} q_*(\lambda) \leq \sum_{\lambda} q_*^2(\lambda) + \frac{a^6}{p}. \quad (76)$$

Now for any  $\tau \geq 1$  consider the set

$$S_{\tau} := \{\lambda \neq 0, 1 : |q_*(\lambda)| \geq \tau\}.$$

Clearly, for an arbitrary  $\lambda \neq 0$  the number  $q(\lambda)$  is

$$\begin{aligned} q(\lambda) &:= |\{a_1, a_2, a \in A : a_1 - \lambda a_2 + (\lambda - 1)a = 0, a_2 \neq a\}| = \\ &= |\{a_1, a_2, a \in A : a_1 - \lambda a_2 + (\lambda - 1)a = 0\}| + 1. \end{aligned}$$

Applying the Fourier transform, we get

$$q_*(\lambda) = p^{-1} \sum_{r \neq 0} \widehat{A}(r) \widehat{A}(-\lambda r) \widehat{A}((\lambda - 1)r) + 1. \quad (77)$$

In particular, at least  $q_*(\lambda)/2$  of the mass of  $q_*(\lambda)$  is contained in the set of non-zero  $r$ ,  $r \in \text{Spec}_{\varepsilon}(A) \cap \lambda^{-1} \text{Spec}_{\varepsilon}(A) \cap (\lambda - 1)^{-1} \text{Spec}_{\varepsilon}(A)$ , where  $\varepsilon = q_*(\lambda)/(8|A|^2)$ . Thus we have obtained

$|S_\tau|$  linear equations of the form (12). Also, it is easy to see that we have for the correspondent system  $\mathcal{E}$  that  $\mathcal{T}(\mathcal{E}) = |S_\tau|$ . Using the arguments and the notations of the proof of the second part of Theorem 28, we constructing the sets  $W_1, W_2$  and the numbers  $\Delta_1, \Delta_2$  such that

$$|S_\tau| \tau^2 p^2 \lesssim |A| p \Delta_1^2 \Delta_2^2 \sum_r W_2(r) (W_1 \otimes S_\tau^{-1})(r).$$

Applying the first part of Proposition 27 with  $B = W_1$ ,  $C = W_2^{-1}$  and  $D = S_\tau$  as well as the Parseval identity, we obtain

$$\begin{aligned} |S_\tau| \tau^2 p^2 &\lesssim a p \Delta_1^2 \Delta_2^2 \delta^{-1/3} (a/\Delta_1)^{2/3} (a/\Delta_2)^{2/3} |W_1|^{1/2} |W_2|^{1/2} |S_\tau|^{1/2} = \\ &= a^{7/3} p \Delta_1^{4/3} \Delta_2^{4/3} \delta^{-1/3} |W_1|^{1/2} |W_2|^{1/2} |S_\tau|^{1/2} \leq a^{10/3} p^2 \Delta_1^{1/3} \Delta_2^{1/3} \delta^{-1/3} |S_\tau|^{1/2} \leq \\ &\leq a^{10/3} p^2 (\|\widehat{A}\|'_\infty)^{2/3} \delta^{-1/3} |S_\tau|^{1/2} \leq \delta^{11/3} p^6 |S_\tau|^{1/2} \end{aligned} \quad (78)$$

or, in other words,

$$|S_\tau| \lesssim p^8 \delta^{22/3} \tau^{-4} \quad (79)$$

provided the following conditions hold

$$|W_1| < \delta^{-1/6} (a/\Delta_1)^{2/3} \sqrt{p}, \quad |W_2| < \delta^{-1/6} (a/\Delta_2)^{2/3} \sqrt{p}. \quad (80)$$

Let us check conditions (80) later. In view of inequality (75), it follows that

$$\sum_{\lambda \neq 0,1} q_*^2(\lambda) \lesssim p^8 \delta^{22/3} \tau^{-2} + \tau \sum_{\lambda} |q_*(\lambda)| \leq p^8 \delta^{22/3} \tau^{-2} + 2\tau \delta^3 p^3.$$

The optimal choice of  $\tau$  is  $\tau = \tau_0 \sim \delta^{13/9} p^{5/3} = a^{13/9} p^{2/9}$ . Thus

$$\sum_{\lambda \neq 0,1} q_*^2(\lambda) \lesssim \tau_0 a^3 \ll a^{40/9} p^{2/9}.$$

Returning to (76) and using

$$-a \leq q_*(0) = q_*(1) = a^2 - a - \frac{a^3}{p} \leq a^2,$$

we obtain

$$|\mathbb{T}(A) - \frac{a^6}{p}| \lesssim a^4 + a^{40/9} p^{2/9} \lesssim a^{40/9} p^{2/9}$$

as required.

It remains to insure that conditions (80) takes place and it is sufficient to check them for  $\tau \geq \tau_0$ . Put  $\varepsilon_1 = \Delta_1/|A|$ ,  $\varepsilon_2 = \Delta_2/|A|$ . Further it is easy to see that  $\varepsilon_1, \varepsilon_2 \gg \tau/a^2 \geq \tau_0/a^2$  and hence (80) is a consequence of the Parseval identity and the following estimates

$$|W_1| \leq p/(a\varepsilon_1^2) < \delta^{-1/6} \varepsilon_1^{-2/3} \sqrt{p}$$

or, in other words,

$$\delta^{1/6} \sqrt{p} = a^{1/6} p^{1/3} \ll a(a^{-5/9} p^{2/9})^{4/3} = a(\tau_0/a^2)^{4/3} < a\varepsilon_1^{4/3}. \quad (81)$$

The first inequality in (81) follows from the condition  $a > p^{2/5}$ . Similar bound takes place for the set  $W_2$ . This completes the proof.  $\square$

Of course, estimate (74) is an asymptotic formula just for sets  $A$  with  $|A| \gtrsim p^{11/14}$ . For sets  $A$ , having the medium size  $p^{2/5} < |A| \lesssim p^{11/14}$  inequality (74) is just a non-trivial upper bound for the quantity  $\mathsf{T}(A)$ . Also, notice that one can improve bound (74), using knowledge about  $\|\widehat{A}\|'_\infty$ , see estimate (78).

Using formulas (71), (72), (73) and the Cauchy–Schwarz inequality, we obtain

**Corollary 34** *Suppose  $A \subseteq \mathbb{F}_p$  such that  $|A| \gtrsim p^{11/14}$ . Then*

$$|R[A]| \geq (1 - o(1))p.$$

The last application of this section concerns mixed energies of a set.

In [10], see Lemma 21, developing the investigations from [6] (see Theorem 2 from here), authors obtained a sum–product result for sets  $A$ ,  $p^{1/2} < |A| \leq p^{2/3}$ , namely

**Lemma 35** *Let  $A \subseteq \mathbb{F}_p$ ,  $X \subseteq \mathbb{F}_p^*$ . Suppose  $|X| = O(|A|^2)$  and  $|A|^2|X| = O(p^2)$ . Then*

$$\sum_{x \in X} \mathsf{E}^+(A, xA) \ll \mathsf{E}^+(A)^{1/2} |A|^{3/2} |X|^{3/4}.$$

It is easy to see that the method of the proof of Theorem 28 allows to obtain a similar result in the regime of large sets  $A$ .

**Theorem 36** *Let  $A \subseteq \mathbb{F}_p$  be a nonempty set,  $\delta = |A|/p$ . Then for any  $X \subseteq \mathbb{F}_p^*$ ,*

$$|X| \leq |A|^{1/2} p^{-1/3} (\|\widehat{A}\|'_\infty)^{4/3} \quad (82)$$

one has

$$\left| \sum_{x \in X} \mathsf{E}^+(A, xA) - \frac{|X||A|^4}{p} \right| \lesssim \log^C(p/|A|) \cdot \delta^{8/3} |X|^{1/2} p^3 \cdot \left( \frac{\|\widehat{A}\|'_\infty}{|A|} \right)^{2/3}, \quad (83)$$

where  $C > 0$  is an absolute constant. In particular, for any such  $X$  the following holds

$$\left| \sum_{x \in X} \mathsf{E}^+(A, xA) - \frac{|X||A|^4}{p} \right| \lesssim \delta^{8/3} |X|^{1/2} p^3. \quad (84)$$

Proof. Using formula (11), we get as in the proof of Theorem 28

$$\begin{aligned}\sigma &:= \sum_{x \in X} \mathbb{E}^+(A, xA) = \frac{|X||A|^4}{p} + \frac{1}{p} \sum_{x \in X} \sum_{r \neq 0} |\hat{A}(r)|^2 |\hat{A}(xr)|^2 = \\ &= \frac{|X||A|^4}{p} + \theta |X| \varepsilon^2 |A|^3 + \frac{1}{p} \sum_{x \in X} \sum_{r \in B} |\hat{A}(r)|^2 |\hat{A}(xr)|^2 = \frac{|X||A|^4}{p} + \theta |X| \varepsilon^2 |A|^3 + \sigma_1, \quad (85)\end{aligned}$$

where  $|\theta| \leq 1$ ,  $B = \text{Spec}_\varepsilon(A) \setminus \{0\}$  and  $\varepsilon$  is a parameter,

$$\varepsilon^2 \sim (\|\hat{A}\|'_\infty)^{2/3} |X|^{-1/2} \delta^{-1} p^{-2/3}. \quad (86)$$

Further applying bound (22) and our assumption, we obtain

$$\delta (\|\hat{A}\|'_\infty)^{8/3} p^{1/3} = ap^{-2/3} (\|\hat{A}\|'_\infty)^{8/3} \gtrsim |X|^2.$$

and hence

$$|B| \leq \frac{1}{\delta \varepsilon^2} \lesssim \delta^{-1/6} \varepsilon^{-2/3} \sqrt{p}. \quad (87)$$

Now using the pigeonholing principle twice, we find two numbers  $\Delta_1, \Delta_2$  and two sets  $W_1, W_2 \subseteq B$  such that

$$p\sigma_1 \lesssim \Delta_1^2 \Delta_2^2 \cdot \sum_{x \in X} \sum_r W_1(r) W_2(xr) = \Delta_1^2 \Delta_2^2 \cdot \sum_{x \in X} (W_1^{-1} \otimes W_2)(x)$$

and  $\Delta_1 < |\hat{A}(r)| \leq 2\Delta_1$  for  $r \in W_1$ ,  $\Delta_2 < |\hat{A}(r)| \leq 2\Delta_2$  for  $r \in W_2$ . Applying Parseval identity (6), we get

$$\Delta_1^2 |W_1| \leq |A|p, \quad \Delta_2^2 |W_2| \leq |A|p. \quad (88)$$

By (87), we have  $|W_1|, |W_2| \leq |B| < \delta^{-1/6} \varepsilon^{-2/3} \sqrt{p}$ . Using the first part of Proposition 27 as well as Theorem 20 and formula (88), we obtain

$$\begin{aligned}p\sigma_1 &\lesssim \Delta_1^2 \Delta_2^2 |X|^{1/2} |W_1|^{1/2} |W_2|^{1/2} \delta^{-1/3} (\Delta_1/|A|)^{-2/3} (\Delta_2/|A|)^{-2/3} = \\ &= \Delta_1^{4/3} \Delta_2^{4/3} |A|^{4/3} |X|^{1/2} |W_1|^{1/2} |W_2|^{1/2} \delta^{-1/3} \leq \Delta_1^{1/3} \Delta_2^{1/3} \delta^2 p^{10/3} |X|^{1/2}.\end{aligned}$$

Using trivial bounds  $\Delta_1, \Delta_2 \leq \|\hat{A}\|'_\infty$  and our choice (86) of the parameter  $\varepsilon$ , we obtain, returning to (85) that

$$\begin{aligned}\left| \sigma - \frac{|X||A|^4}{p} \right| &\lesssim |X| \varepsilon^2 \delta^3 p^3 + \delta^2 (\|\hat{A}\|'_\infty)^{2/3} |X|^{1/2} p^{7/3} \ll \delta^2 (\|\hat{A}\|'_\infty)^{2/3} |X|^{1/2} p^{7/3} = \\ &= \delta^{8/3} |X|^{1/2} p^3 \cdot \left( \frac{\|\hat{A}\|'_\infty}{|A|} \right)^{2/3}.\end{aligned}$$

This completes the proof.  $\square$

For example, if  $|A| \leq p/2$  then by Parseval identity (6), we get  $\|\widehat{A}\|'_\infty \gg |A|^{1/2}$  and hence condition (82) satisfies if  $|X| \leq |A|^{7/6} p^{-1/3}$ .

Notice that in bound (84) the term  $|X||A|^4/p$  dominates if  $|X| \gtrsim \delta^{-8/3}$ . On the other hand, in view of trivial bound

$$\left| \sum_{x \in X} \mathbb{E}^+(A, xA) - \frac{|X||A|^4}{p} \right| < |A|^2 p$$

which follows from formula (85), we see that Theorem 36 has sense for sets  $A$  with small  $\|\widehat{A}\|'_\infty$  only.

## References

- [1] E. AKSOY YAZICI, B. MURPHY, M. RUDNEV, I.D. SHKREDOV, *Growth estimates in positive characteristic via collisions*, arXiv:1512.06613v1 [math.CO] 21 Dec 2015.
- [2] N. ALON, J. BOURGAIN, *Additive Patterns in Multiplicative Subgroups*, Geom. Funct. Anal. **24:3** (2014), 721–739.
- [3] T. F. BLOOM, *A quantitative improvement for Roth’s theorem on arithmetic progressions*, doi: 10.1112/jlms/jdw010.
- [4] P. CANDELA, O. SISASK, *On the asymptotic maximal density of a set avoiding solutions to linear equations modulo a prime*, Acta Math. Hungar. **132:3** (2011), 223–243.
- [5] S. V. KONYAGIN, I. D. SHKREDOV, *New results on sum-products in  $\mathbb{R}$* , Transactions of Steklov Mathematical Institute **294** (2016), 87–98.
- [6] G. PETRIDIS, *Products of Differences in Prime Order Finite Fields*, arXiv:1602.02142 [math.CO] 5 Feb 2016.
- [7] O. ROCHE-NEWTON, M. RUDNEV, I. D. SHKREDOV, *New sum-product type estimates over finite fields*, Adv. Math. **293** (2016), 589–605.
- [8] W. RUDIN, *Fourier analysis on groups*, Wiley 1990 (reprint of the 1962 original).
- [9] M. RUDNEV, *On the number of incidences between planes and points in three dimensions*, To appear in Combinatorica, arXiv:1407.0426v5 [math.CO] 5 Dec 2015.
- [10] M. RUDNEV, I. D. SHKREDOV, S. STEVENS, *On the energy variant of the sum-product conjecture*, arXiv:1607.05053v2 [math.CO] 21 Jul 2016.
- [11] I. Z. RUZSA, *Solving a linear equation in a set of integers I*, Acta Arith. **65** (1993), 259–282.
- [12] I. Z. RUZSA, *Solving a linear equation in a set of integers II*, Acta Arith. **72** (1995), 385–397.
- [13] I. Z. RUZSA, *Arithmetical progressions and the number of sums*, Periodica Math. Hung. **25** (1992) 105–111.

- [14] T. SANDERS, *On Roth's theorem on progressions*, Ann. of Math. **174** (2011), 619–636.
- [15] A. SÁRKÖZY, *On sums and products of residues modulo  $p$* , Acta Arithm. **118** (2005), 403–409.
- [16] Y. V. STANCHESCU, *Planar sets containing no three collinear points and non-averaging sets of integers*, Discrete Mathematics **256** (2002), 387–395.
- [17] I. D. SHKREDOV, *On Sets of Large Exponential Sums*, Izvestiya of Russian Academy of Sciences, **72**:1 (2008), 161–182.
- [18] I. D. SHKREDOV, *On sumsets of dissociated sets*, Online Journal of Analytic Combinatorics, **4** (2009), 1–26.
- [19] I. D. SHKREDOV, *Energies and structure of additive sets*, Electronic Journal of Combinatorics, **21**:3 (2014), #P3.44, 1–53.
- [20] I. D. SHKREDOV, *Difference sets are not multiplicatively closed*, arXiv:1602.02360, 2016.
- [21] I. D. SHKREDOV, D. ZHELEZOV, *On additive bases of sets with small product set*, arXiv:1606.02320v1 [math.NT] 7 Jun 2016.
- [22] T. TAO, V. VU, *Additive combinatorics*, Cambridge University Press 2006.
- [23] S. YEKHANIN, *A note on plane pointless curves*, Finite Fields Appl. **24** (2006), 418–422.

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